ELEMENTS

OF

DESCRIPTIVE GEOMETRY

WITH APPLICATIONS TO

SPHERICAL AND ISOMETRIC PROJECTIONS, SHADES AND SHADOWS, AND PERSPECTIVE

BY

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NEW YORK .:. CINCINNATI .:. CHICAGO

AMERICAN BOOK COMPANY

O.-B. DESCR. GEOM.
W. P. 6

PREFACE

CHURCH'S "Elements of Descriptive Geometry" was originally published in 1864. The preface to the first edition states: "Without any effort to enlarge or originate, the author has striven to give, with a natural arrangement and in clear and concise language, the elementary principles and propositions of this branch of science, of so much interest to the mathematical student, and so necessary to both the civil and military engineer."

Professor Church succeeded so well in his efforts to produce a practical and well-adapted treatise that it has continued in use as a text-book for more than forty years in the United States Military Academy and in many other academies, technical schools, and colleges. This long-continued use of the book speaks well for its high intrinsic excellence.

During the last few years, however, there have taken place many changes in the methods of teaching the subject, and in the problems required. To meet these new demands the present volume is issued. In its preparation much of Professor Church's text has been used, and his concise and lucid style has been preserved.

Among the salient features of the present work are the following:

The *figures* and *text* are included in the same volume.

General cases are preferred to special ones.

A sufficient number of problems are solved in the *third angle* to familiarize the student with its use.

A treatment of the *profile plane* of projection is introduced.

Many *exercises for practice* have been introduced.

Several *new problems* have been added.

The *old figures have been redrawn*, and many of them have been improved.

Several of the more difficult elementary problems have been illustrated by *pictorial views*.

In the treatment of *curved surfaces*, all problems relating to single-curved surfaces are taken up first, then those relating to warped surfaces, and finally those relating to surfaces of revolution. Experience proves this order to be a logical one, as we here proceed "from the simple to the more complex." Also the student is more quickly prepared for drawing-room work on intersections and developments; and in case it is desired to abbreviate the course by omitting warped surfaces, the remaining problems will be found to be consecutively arranged.

The writer here wishes to acknowledge his indebtedness to the many teachers who have aided him with valuable advice and suggestions in relation to this work. In particular his thanks are due to his esteemed colleagues, Professor H. J. Goulding and Mr. D. E. Foster of the University of Michigan, for their careful reading and correction of the manuscript.

G. M. B.

MAY 14, 1910.

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PART I

ORTHOGRAPHIC PROJECTIONS

Preliminary Definitions

1. Geometry enables us to determine unknown magnitudes, relationships, and forms from those which are known. There are in general two methods of solution for any given problem; namely, the *analytical* and the *graphical*. In the former we arrive at our results by calculation; in the latter we make drawings which represent graphically the true relationships between the points, lines, and surfaces under consideration, and arrive at our results without calculation.

2. Graphics. If the problem relates to points and lines lying in only one plane, the graphical solution may be reached by a simple application of the principles of Geometrical Drawing, or *Plane Graphics.*

If the problem relates to magnitudes not in the same plane, the graphical solution would require an application of the principles of Descriptive Geometry, or the *Graphics of Space.*

3. Descriptive Geometry is that branch of Mathematics which has for its object the explanation of the methods of representing by drawings:

First. All geometrical magnitudes.

Second. The solution of problems relating to these magnitudes in space.

These drawings are so made as to present to the eye, situated at a particular point, the same appearance as the magnitude or object itself, were it placed in the proper position.

The representations thus made are the *projections* of the magnitude or object.

The planes upon which these projections are usually made are the *planes of projection.*

The point at which the eye is situated is the *point of sight.*

4. Projections. When the point of sight is in a perpendicular drawn to the plane of projection through any point of the drawing, and at an infinite distance from this plane, the projection is *orthographic.*

When the point of sight is within a finite distance of the drawing, the projection is *scenographic*, and is commonly called the *perspective* of the magnitude or object.

If a straight line be drawn through a given point and the point of sight, the point in which this line pierces the plane of projection will present to the eye the same appearance as the point itself, and will therefore be the projection of the point on this plane. The line thus drawn is the *projecting line of the point.*

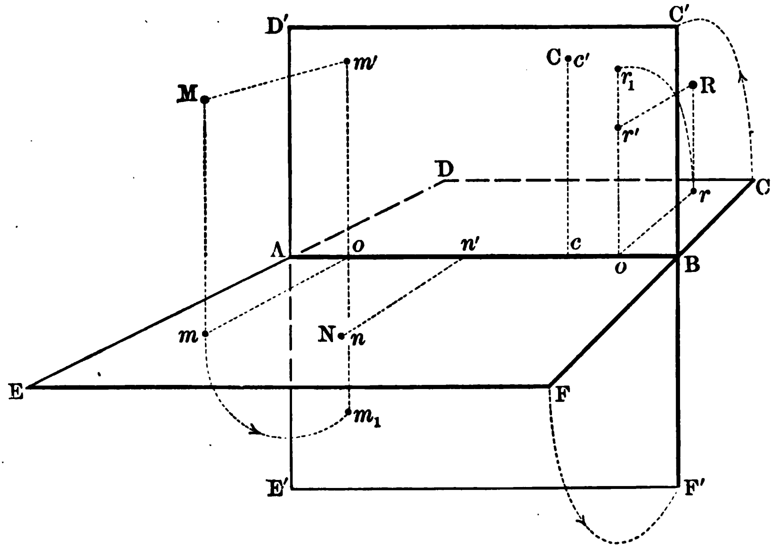


FIG. 1.

5. In the orthographic projection, since the point of sight is at an infinite distance, the projecting lines drawn from any points of an object of finite magnitude to this point, *will be parallel to each other and perpendicular to the plane of projection.*

In this projection two planes are used, at right angles to each other, the one *horizontal* and the other *vertical*, called respectively the *horizontal* and the *vertical plane of projection.*

In Fig. 1, let the planes represented by CDEF and C′D′E′F′ be the two planes of projection, the first the *horizontal* and the second the *vertical.*

Their line of intersection AB is the *ground line.*

These planes form by their intersection four dihedral angles. The *first angle* is above the horizontal and in front of the vertical plane. The *second* is above the horizontal and behind the vertical. The *third* is below the horizontal and behind the vertical. The *fourth* is below the horizontal and in front of the vertical.

Representation of Points, Lines, and Planes

6. Representation of points. Let M, Fig. 1, be any point in space. Through it draw Mm perpendicular to the horizontal, and Mm' perpendicular to the vertical plane; m will be the projection of M on the horizontal, and m' that on the vertical plane (Art. 5). Hence, the *horizontal projection of a point* is the foot of a perpendicular through the point to the horizontal plane; and the *vertical projection of a point* is the foot of a perpendicular through it to the vertical plane.

The lines Mm and Mm' are the horizontal and vertical *projecting lines* of the point.

7. Proposition I.* **The distance of a point from the horizontal plane is equal to the distance of its vertical projection from**

* It is important that the student should be able to state and prove all propositions given in this book.

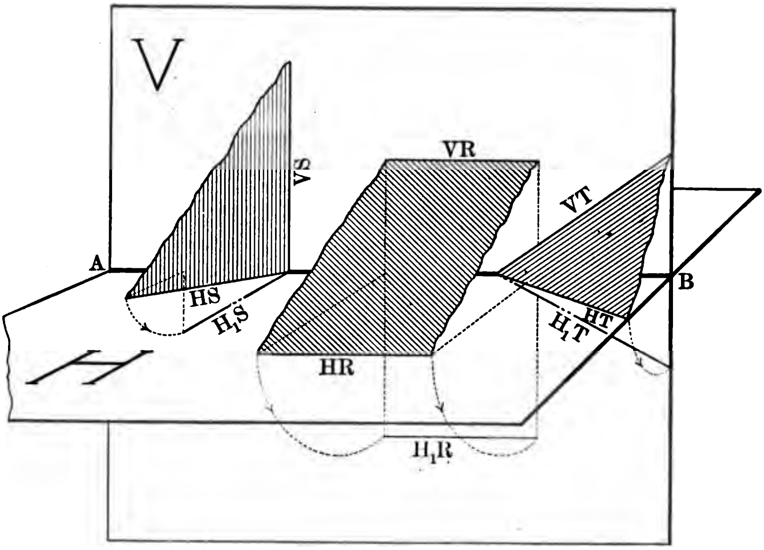


FIG. 2.

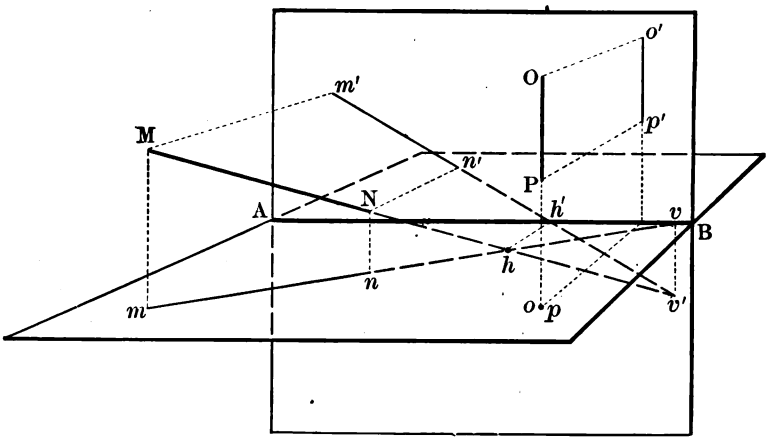


FIG. 3.

the ground line ; and the distance of the point from the vertical plane is equal to that of its horizontal projection from the ground line.

Through the lines Mm and Mm', Fig. 1, pass a plane. It will be perpendicular to both planes of projection, since it contains a straight line perpendicular to each, and therefore perpendicular to the ground line AB. It intersects these planes in the lines mo and $m'o$, both perpendicular to the ground line at the same point, forming the rectangle Mo. By an inspection of the figure it is seen that $Mm = m'o$, and $Mm' = mo$.

8. Proposition II. **If the two projections of a point are given, the point is completely determined.** For if at the horizontal projection m, Fig. 1, a perpendicular be erected to the horizontal plane, it will contain the point M. A perpendicular to the vertical plane at m' will also contain M ; hence the point M is determined by the intersection of these two perpendiculars.

If N be in the horizontal plane, $Nn = 0$, and the point is its own horizontal projection. The vertical projection will be in the ground line at n'. Similarly, if C be in the vertical plane, it will be its own vertical projection, and its horizontal projection will be in the ground line at c.

If the point be in the ground line, it will be its own horizontal and also its own vertical projection.

9. Representation of planes. Let HT–VT, Fig. 2, be a plane, oblique to the ground line, intersecting the planes of projection in the lines HT and VT respectively. Its intersection with the horizontal plane is the *horizontal trace* of the plane, and its intersection with the vertical plane is the *vertical trace*.

10. Proposition III. **If the two traces of a plane are given, the plane is completely determined.** Why?

11. Representation of straight lines. Let MN, Fig. 3, be any straight line in space. Through it pass a plane Mmn per-

pendicular to the horizontal plane; *mn* will be the horizontal, and *vv′*, perpendicular to the ground line, the vertical trace of this plane. Also through MN pass a plane M*m′n′* perpendicular to the vertical plane; *m′n′* will be its vertical, and *h′h* its horizontal trace. The traces *mn* and *m′n′* are the projections of the line, the points *m*, *m′*, *n*, *n′* being the projections of the extremities of the line. Hence, the *horizontal projection* of a straight line coincides with the *horizontal trace of a plane passed through the line perpendicular to the horizontal plane;* and the *vertical projection* of a straight line coincides with the *vertical trace of a plane through the line perpendicular to the vertical plane.*

The planes M*mn* and M*m′n′* are respectively the *horizontal* and the *vertical projecting planes* of the line.

12. PROPOSITION IV. **The two projections of a straight line being given, the line will in general be completely determined**; for if through the horizontal projection we pass a plane perpendicular to the horizontal plane, it will contain the line; and if through the vertical projection we pass a plane perpendicular to the vertical plane, it will also contain the line. The intersection of these planes must, therefore, be the line. Hence we say *a straight line is given by its projections.*

13. The projections *mn* and *m′n′* are also manifestly made up of the projections of all the points of the line MN. Hence:

PROPOSITION V. **If a straight line pass through a point in space, its projections will pass through the projections of the point.** Likewise,

PROPOSITION VI. **If, in either plane, the projections of any two points of a straight line are given, the straight line joining these projections will be the projection of the given line.**

14. Propositions relating to the Point, Line, and Plane.

PROPOSITION VII. **The two traces of a plane must intersect the ground line at the same point.** For if they should intersect

it at different points, the plane would intersect it in two points, which is impossible.

PROPOSITION VIII. **If a plane is parallel to the ground line, its traces must be parallel to the ground line** (Fig. 2). For if they are not parallel, they must intersect it; in which case the plane would have at least one point in common with the ground line, which is contrary to the hypothesis.

PROPOSITION IX. **If a plane is parallel to either plane of projection, it will have but one trace, which will be on the other plane, and parallel to the ground line.**

PROPOSITION X. **If a plane is perpendicular to the horizontal plane, its vertical trace will be perpendicular to the ground line,** as VS in Fig. 2. For the vertical plane is also perpendicular to the horizontal plane; hence the intersection of the two planes, which is the *vertical trace*, must be perpendicular to the horizontal plane, and therefore to the ground line which intersects it.

Likewise if a plane is perpendicular to the vertical plane, its horizontal trace will be perpendicular to the ground line.

PROPOSITION XI. **If a plane simply pass through the ground line, its position is not determined.**

PROPOSITION XII. **If two planes are parallel, their traces on the same plane of projection are parallel, for these traces are the intersections of the parallel planes by a third plane.**

But if two planes are *perpendicular* to each other, their traces on the same plane will *not* in general be perpendicular. Will they ever be? If so, when?

PROPOSITION XIII. **If a straight line is perpendicular to either plane of projection, its projection on that plane will be a point, and its projection on the other plane will be perpendicular to the ground line.** See OP, Fig. 3.

PROPOSITION XIV. **If a line is parallel to either plane of projection, its projection on that plane will evidently be parallel**

and equal to the line itself, and its projection on the other plane will be parallel to the ground line. See Fig. 4.

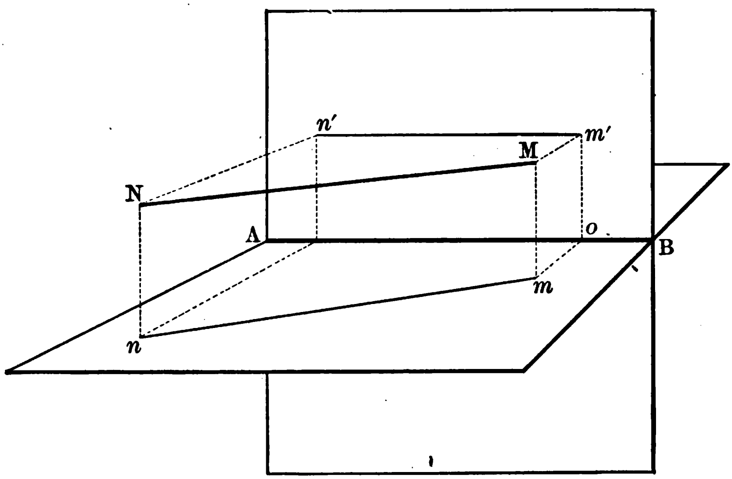


Fig. 4.

Also, if a straight line is parallel to both planes of projection or to the ground line, both projections will be parallel to the ground line.

Proposition XV. **If a line lies in either plane of projection, its projection on that plane will be the line itself, and its projection on the other plane will be in the ground line.** Thus in Fig. 5, RS in the vertical plane is its own vertical projection, and *rs* in the ground line is its horizontal projection.

Proposition XVI. **If the two projections of an unlimited straight line are perpendicular to the ground line, the line is undetermined,** as the two projecting planes coincide, forming only one plane, and do not by their intersection determine the line as in Art. 12.

All unlimited lines in this plane will have the same projections. Thus, in Fig. 5 *mn* and *m'n'* are both perpendicular to

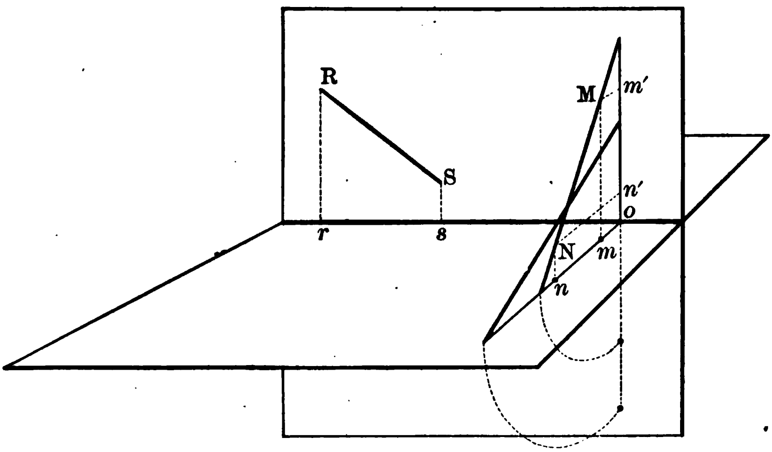


FIG. 5.

the ground line; and any line in the plane MN*o* will have these for its projections.

PROPOSITION XVII. **If, however, the projections of two points of the line are given, the line will then be determined;** that is, if (m, m') and (n, n') are given, the two points, M and N, will be determined, and, of course, the straight line which joins them.

PROPOSITION XVIII. **All lines and points situated in a plane perpendicular to either plane of projection, will be projected on this plane in the corresponding trace of the plane.**

PROPOSITION XIX. **If two straight lines are parallel, their projections on the same plane will be parallel.** For their projecting planes are parallel, since they contain parallel lines and are perpendicular to the same plane; hence their traces will be parallel (Prop. XII), and these traces are the projections.

If two lines are *perpendicular* to each other, their projections on the same plane will *not* in general be perpendicular; but,

PROPOSITION XX. **If two straight lines are perpendicular to each other, and one of them is parallel to one of the planes**

of projection, their projections on that plane will be perpendicular. For the projecting plane of the line which is not parallel to the plane of projection is perpendicular to the second line, and also to its projection, since this projection is parallel to the line itself (Prop. XIV); and since this projection is perpendicular to this projecting plane, it is perpendicular to the trace of this projecting plane, which is the projection of the first line.

PROPOSITION XXI. **Every straight line of a plane, not parallel to the horizontal plane of projection, will pierce it in the horizontal trace of the plane, and if not parallel to the vertical plane, will pierce it in the vertical trace.** Why?

PROPOSITION XXII. **If a straight line lies in a plane and is parallel to the horizontal plane, its horizontal projection will be parallel to the horizontal trace of the plane, and its vertical projection will be parallel to the ground line.** Prove this.

What will be true of the projections of a straight line that is parallel to the vertical plane?

PROPOSITION XXIII. **If two intersecting lines are each parallel to the same plane of projection, the angle between their projections on that plane will be equal to the angle between the lines.** Prove this.

PROPOSITION XXIV. **If a straight line is perpendicular to a plane, its projections will be respectively perpendicular to the traces of the plane.** For if the line PQ, Fig. 6, is perpendicular to the plane MR, the horizontal projecting plane of the line is perpendicular to the given plane MR, since it contains a line, PQ, perpendicular to it. This projecting plane is also perpendicular to the horizontal plane (Art. 11). It is therefore perpendicular to the intersection SR, of these two planes, which is the horizontal trace of the given plane. Hence *pq*, the *horizontal projection of the line*, which is a line of this projecting plane, must be perpendicular *to the horizontal trace.*

In the same way it may be proved that the *vertical projection* of the line will be perpendicular *to the vertical trace.*

PROPOSITION XXV. *Conversely,* **If the projections of a straight line are respectively perpendicular to the traces of a plane, the line will be perpendicular to the plane.** For if through the horizontal projection of the line its horizontal projecting plane be passed, it will be perpendicular to the horizontal trace of the given plane, and therefore perpendicular to the plane. In the same way it may be proved that the vertical projecting plane of the line is perpendicular to the given plane; therefore the intersection of these two planes, which is the *given line,* is perpendicular to the given plane.

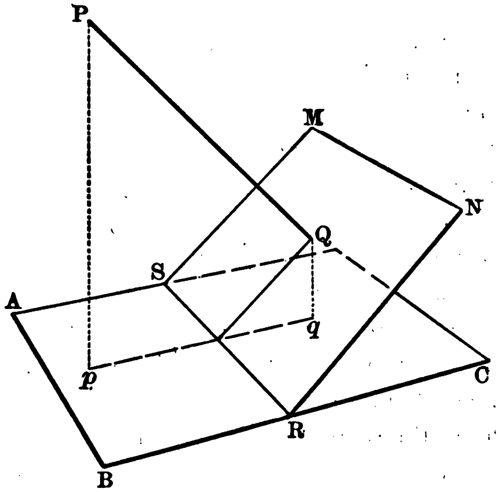


FIG. 6.

Is the above proposition true when the plane is parallel to the ground line?

But if a straight line is *parallel* to a plane, the projections of the line will *not* in general be parallel to the traces of the plane. Will they ever be? If so, under what conditions?

15. Rotation of the horizontal plane. In order to represent both projections of an object on the same sheet of paper or plane, after the projections are made as in the preceding articles, the horizontal plane is rotated about the ground line as an axis until it coincides with the vertical plane, that portion of the horizontal plane which is in front of the ground line falling

below it in the position ABF′E′, Fig. 1, and that part which is back of the ground line coming up in the position ABC′D′.

In this new position of the planes it will be observed that, the planes being regarded as indefinite in extent, all that part of the plane of the paper which is below the ground line will represent not only that part of the vertical plane which is below the ground line, but also that part of the horizontal plane which is in front of the ground line; while the part above the ground line represents that part of the vertical plane which is above the ground line, and also that part of the horizontal plane which is back of the ground line.

16. After the horizontal plane is rotated as in the preceding article, the point *m*, in Fig. 1, will take the position m_1 in the line $m'o$ produced, and the two projections m_1 and m' will then be in the same straight line, perpendicular to the ground line. Hence:

PROPOSITION XXVI. **The two projections of the same point must be in the same straight line, perpendicular to the ground line.**

PROPOSITION XXVII. **If a point is in the first angle, its horizontal projection will be below, and its vertical projection above, the ground line.**

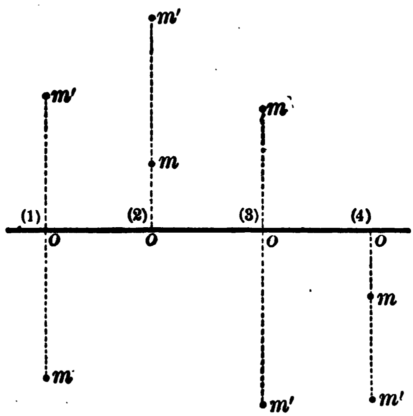


FIG. 7.

A point in the second angle will have both projections above the ground line, as R, Fig. 1.

A point in the third angle will have its horizontal projection above, and its vertical projection below, the ground line. Why?

A point in the fourth angle will have its projections, where?

Conversely, we can tell in what angle a given point lies by noticing the location of its projections with respect to the ground line.

PROPOSITION XXVIII. **If any two lines intersect, the straight line joining the points in which their projections intersect must be perpendicular to the ground line.** Prove this.

17. Notation to be used in the description of drawings. A point in space will be designated by some capital letter, as M. Its horizontal projection will be designated by the corresponding lower-case letter, as *m*; and its vertical projection by the

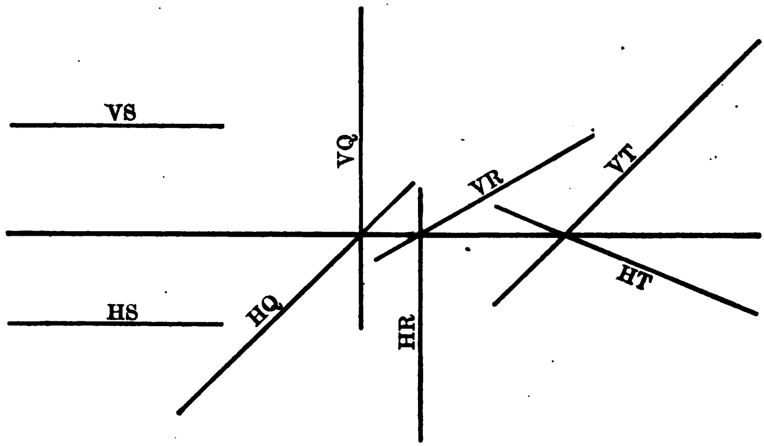


FIG. 8.

same letter *primed*, as m'. We may speak of the point itself either as the point (m, m') or simply as the point M.

A plane will be designated by some capital letter, as S, T, R. Its horizontal trace will be designated, as in Figs. 2 and 8, by a capital H placed before the letter of the plane, as HS, HT, etc. Its vertical trace will be designated by a capital V placed before the letter of the plane, as VS, VT, etc.

Lines given by their projections, as in Fig. 9, will be described as the line (mn, $m'n'$), the letters on the horizontal projection being first in order, or simply the line MN.

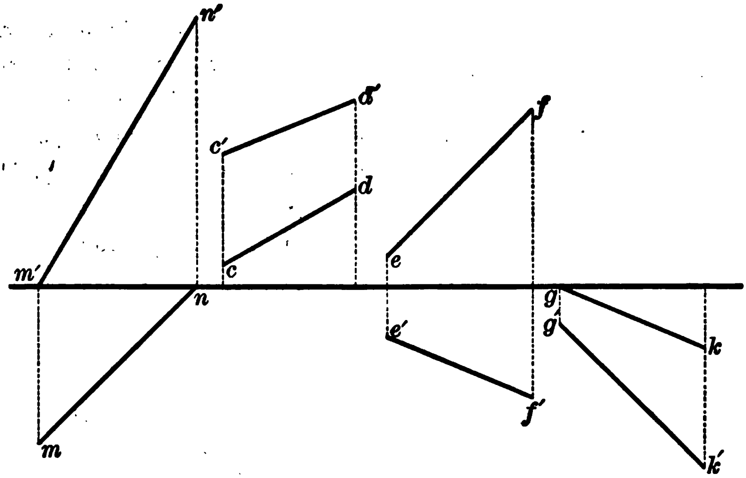


FIG. 9.

The planes of projection will often be described by the capitals H and V; H denoting the horizontal, and V the vertical, plane.

18. The projections of the same point will be connected by a dotted line, thus ....

Traces of planes which are given or required, when they can be seen from the point of sight, — that is, when the view is not obstructed by some intervening opaque object, — are drawn full. When not seen, or when they are the traces of auxiliary planes, not the projecting planes of straight lines, they will be drawn dashed and dotted, thus:

Lines, or portions of lines, either given or required, when seen, will have their projections full. When not seen, or auxiliary,

their projections will be dashed, thus :

— — — — — — — — — — —

In the construction of problems, the planes of projection and all auxiliary surfaces will be regarded as transparent.

All lines or surfaces are regarded as indefinite in extent, unless limited by their form, or a definite portion is considered for a special purpose. Thus the ground line and projections of lines in Fig. 9 are supposed to be produced indefinitely, the lines delineated simply indicating the directions.

Exercises for Practice

Let the following examples be treated in accordance with Propositions I to XXVIII.*

Ex. 1. State in what angle each of the points shown in Fig. 10 is located, and whether the point is nearer H or V.

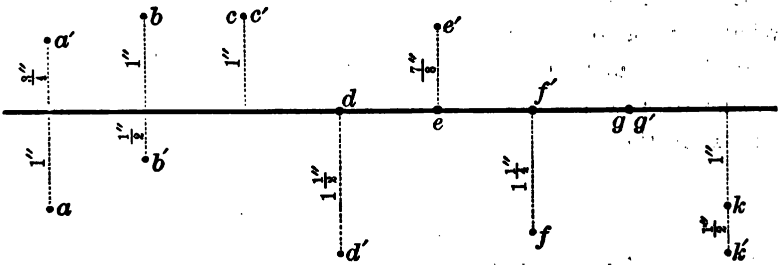


Fig. 10.

Ex. 2. Show the projections of the following points properly lettered and with distances given.

The pt. A, 1″ behind V, 1½″ below H.

The pt. B, lying in V, 1″ below H.

The pt. C, 3″ in front of V, 1″ above **H.**

The pt. D, 1″ behind V, lying in H.

* In these examples, ‖ is to be read *parallel;* ⊥, *perpendicular;* ∠, *angle,* and ″, *inches.* A *line* will be understood to be straight, unless otherwise stated.

The pt. E, 2″ behind V, $1\frac{1}{2}''$ below H.
The pt. F, 1″ in front of V, 1″ below H.
The pt. G, lying in V, 2″ above H.
The pt. J, 1″ in front of V, lying in H.
The pt. K, lying in V, lying in H.
The pt. L, in 3d angle, 1″ from H, 2″ from V.
The pt. M, in 2d angle, 3″ from H, 2″ from V.
The pt. N, in 1st angle, $1\frac{1}{2}''$ from H, $3\frac{1}{2}''$ from V.
The pt. P, in 4th angle, 4″ from H, 1″ from V.

Ex. 3. If the H projection of a line is ‖ to the ground line, what conclusion do you draw (*a*) as to the *position* of the line? (*b*) as to its intersection with the V plane? (*c*) as to the V trace of any plane passed through the line?

Ex. 4. If a straight line lies in a given plane, what conclusion do you draw as to the points where the line pierces the H and V planes respectively?

Ex. 5. If the H trace of a plane is ⊥ to the ground line, what conclusion do you draw (*a*) as to the *position* of the plane? (*b*) as to the ∠ between the plane and the H plane? (*c*) as to the V projection of any line lying in the plane? (*d*) as to the ∠ between the plane and the V plane?

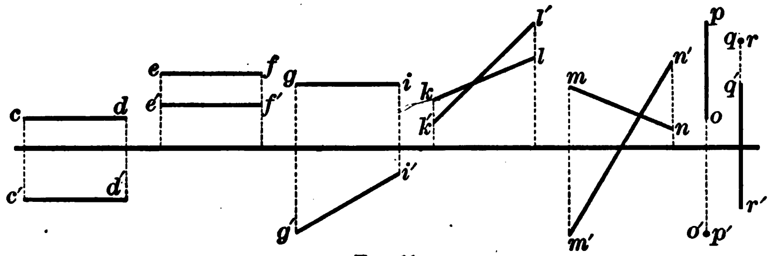


FIG. 11.

Ex. 6. Describe the situation of the lines in Fig. 11 with respect to the *ground line*, the *planes of projection*, and the *angles*.

Ex. 7. Show the projections of the lines

AB, ‖ to H, ‖ to V, in 3d angle.

CD, ‖ to H, ⊥ to V, in 2d angle.

EF, ‖ to H, inclined to V, in 1st angle.

GH, inclined to H, ‖ to V, in 1st angle.

JK, ⊥ to H, ‖ to V, in 2d angle.

MN, inclined to H, inclined to V, in 2d angle.

OP, inclined to both planes of projection and in a plane perpendicular to the ground line, in 1st angle.

QR, inclined to V, and lying in H beyond the ground line.

ST, inclined to H, and lying in V above the ground line.

UV, lying in both H and V.

Ex. 8. Construct the projections of two lines, AB and AC, intersecting in A, one ‖ to H, the other ‖ to V.

Ex. 9. Show the projections of a line joining a point A in the 2d angle with a point B in the 3d angle.

Ex. 10 Show the projections of a line joining a point C in the 4th angle with a point D in the 1st angle.

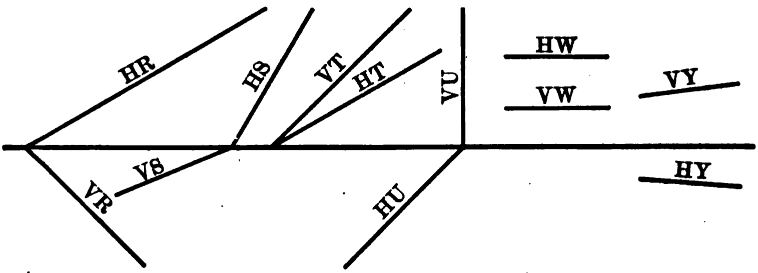


Fig. 12.

Ex. 11. Describe the situation of the planes in Fig. 12 with respect to the *ground line*, the *planes of projection*, and the *angles*.

Ex. 12. Represent by their traces the planes:

A, ⊥ to both H and V.

B, inclined to H; ⊥ to V.

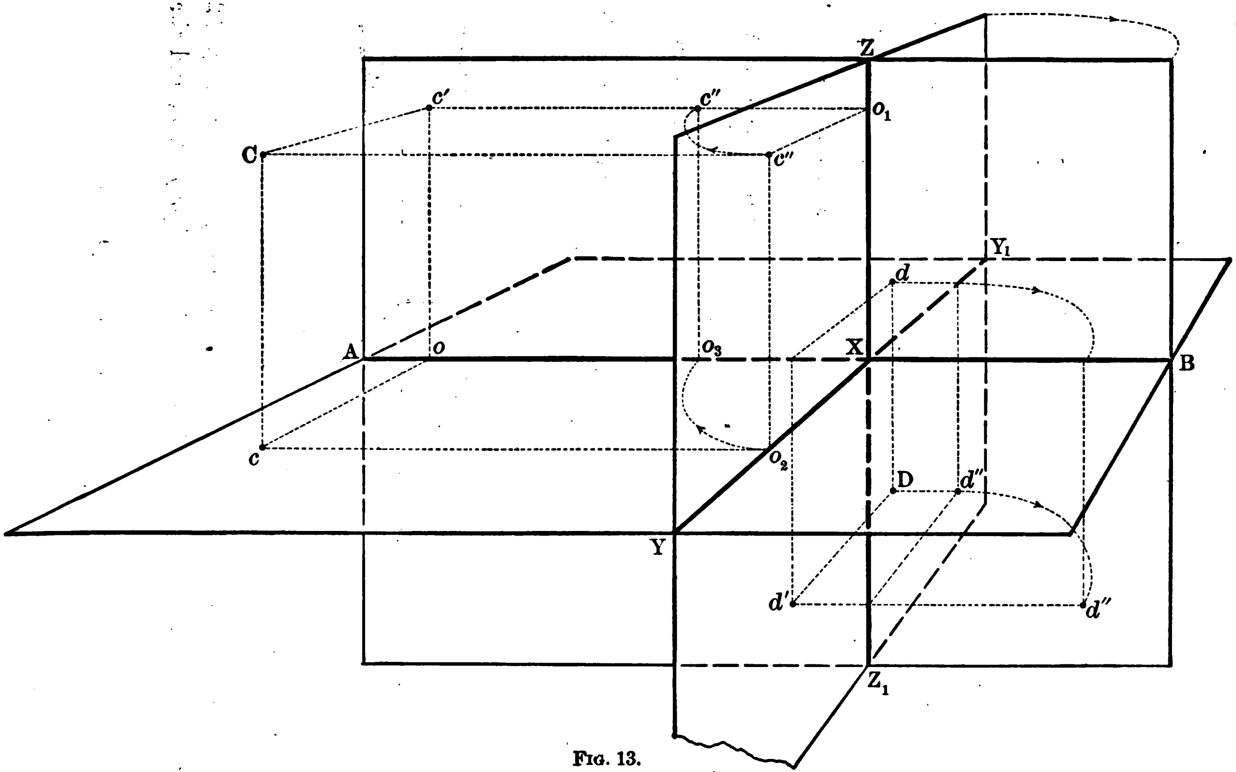


FIG. 13.

C, $\perp$ to H; inclined to V.

D, inclined to both H and V.

E, $\perp$ to H; $\parallel$ to V.

F, $\parallel$ to H; $\perp$ to V.

G, $\parallel$ to the ground line, but not passing through it.

K, containing the ground line.

The Profile Plane of Projection

19. While most problems in descriptive geometry can be solved by means of two planes of projection, it is sometimes convenient to make use of a third plane perpendicular to both H and V, and called the *profile*, or P, *plane of projection.* It is usually placed to the right of the object; and in order to represent on a single sheet of paper all three projections of the object, the profile plane is rotated about its vertical trace as an axis until it coincides with the V plane.

The direction of rotation is usually such as to bring the P and V projections of the object on opposite sides of the profile trace. This practice is almost universal for objects in the *second* or *third angle*, but when objects are placed in the first angle, practice differs. For the purpose of uniformity in the present text, as well as for general convenience, the direction of rotation will be such that the portion lying in front of the V plane falls to the *left* of its V trace, while the portion lying behind the V plane falls to the *right* of the same.

It will be noticed that the ground line and the profile axis cross at right angles and divide the plane of the paper into four *quadrants.*

20. Proposition XXIX. **If a point is in the first angle, its P projection will appear in the upper left-hand quadrant. If a point is in the second angle, its P projection will appear in the upper right-hand quadrant. If in the third angle, its P projection**

appears in the lower right-hand quadrant; and if in the fourth angle, its P projection is in the lower left-hand quadrant.

An examination of Fig. 13 will show the above to be true, and also the following:

21. PROPOSITION XXX. **The distance of the P projection of a point from the ground line is equal to the distance of its V projection from the ground line, and its distance from the profile axis ZZ_1 is equal to the distance of the H projection from the ground line.**

Hence, to determine the profile projection of a point C when its H and V projections are known, we draw through the V projection an indefinite line $c'o_1$ (Figs. 13 and 14) parallel to the ground line. The P projection will lie somewhere in this line. With X as a center and the distance *co* as a radius, draw a quarter circle o_2o_3 in a *clockwise* direction. At the point o_3

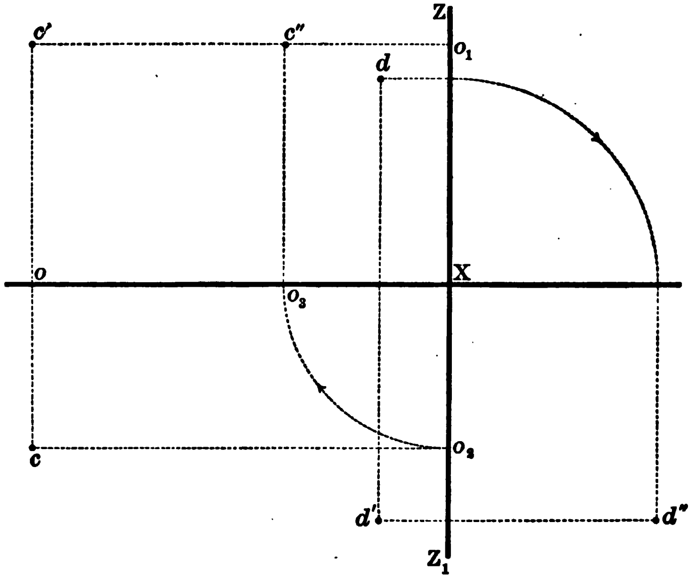


FIG. 14.

erect a perpendicular to the ground line intersecting $c'o_1$ in c'', the required profile projection of C. Similarly for the point D in the third angle.

Ex. 13. Let the student assume points and lines in each of the four angles, and then find their profile projections.

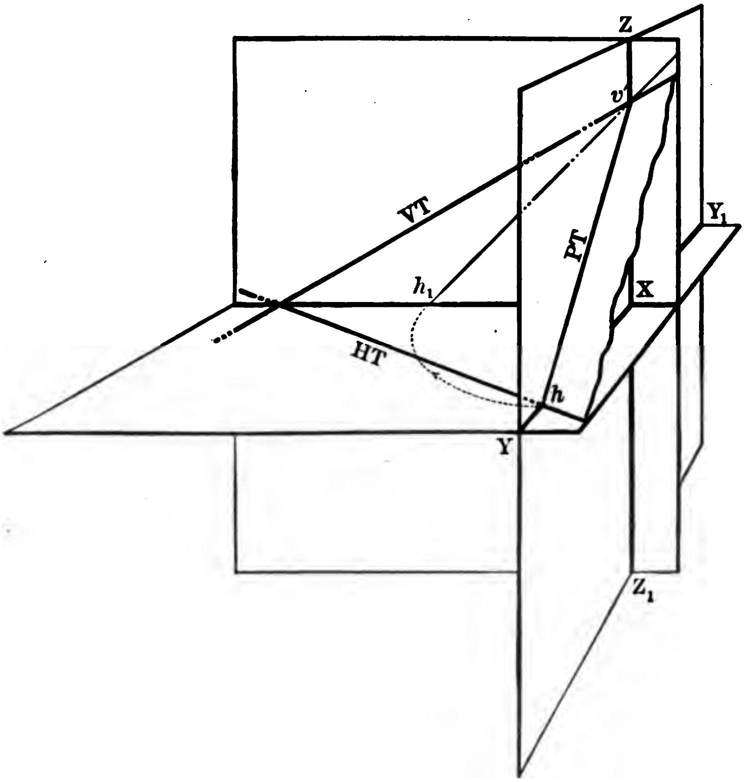


FIG. 15.

22. The profile trace of a plane. In Fig. 15, let T be any oblique plane. Its intersection hv' with the P plane of projection is its profile trace. When the profile plane has been brought into coincidence with V (Art. 19), the point h, where the P and H traces meet, will revolve to the position h_1 in the

ground line. The point v', where the P and V traces meet, will not change. Hence the profile trace will take the position h_1v'.

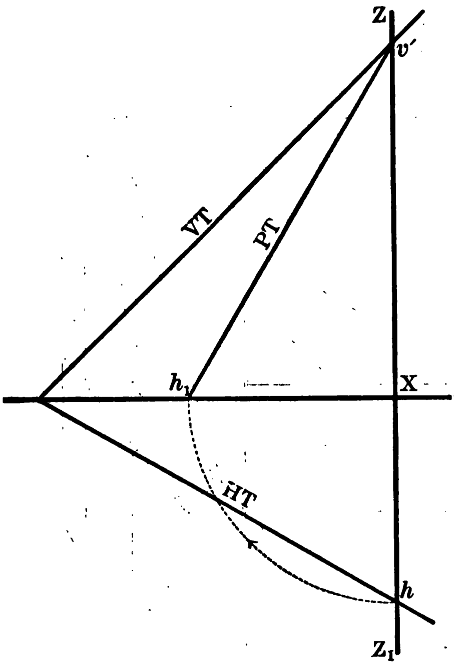


FIG. 16.

To determine the profile trace of a plane, having given the H and V traces, we proceed as follows: With X as center, Fig. 16, and Xh as radius, strike the quarter circle hh_1 in a *clockwise* direction and join h_1 with the point v' where the V trace cuts the profile axis.

Ex. 14. Assume planes such as R, S, T, U, and W, Fig. 12, and find the profile trace of each.

Ex. 15. Assume lines such as GI, KL, QR, and MN, Fig. 11, and find their profile projections.

Construction of Elementary Problems relating to the Point, Line, and Plane

23. Having explained the manner of representing with accuracy points, planes, and straight lines, we are now prepared to represent the solution of a number of important problems relating to these magnitudes in space.

In every problem certain points and magnitudes are given, from which certain other points or magnitudes are to be constructed.

Let a straight line be first drawn on the paper to represent

the ground line; then assume, as in Arts. 16, 17, and 18, the representations of the given objects. The proper solution of the problem will now consist of two distinct parts. The first is a clear statement of the principles and reasoning to be employed in the construction of the drawing. This is the *analysis* of the problem. The second is the construction, in proper order, of the different lines which are used and required in the problem. This is the *construction* of the problem.

24. Problem 1. **To find the points in which a given straight line pierces the planes of projection.**

Let AB, Fig. 17, be the ground line, and (mn, $m'n'$), or simply MN, the given line.

First. To find the point in which this line pierces the horizontal plane.

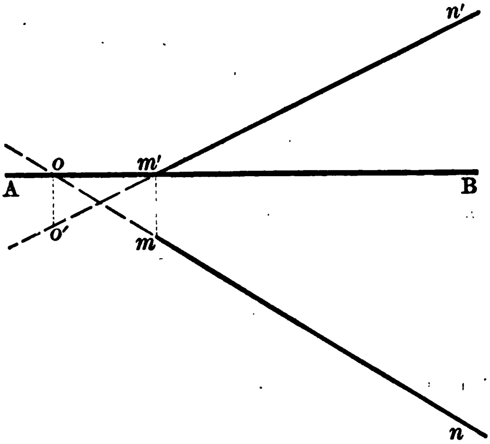


Fig. 17.

Analysis. Since the required point is in the horizontal plane, its vertical projection is in the ground line (Art. 8); and since the point is in the given line, its vertical projection will be in the vertical projection of this line (Prop. V, Art. 13); hence it must be at the intersection of this vertical projection with the ground line. The horizontal projection of the required point must be in a straight line drawn through its vertical projection, perpendicular to the ground line (Prop. XXVI, Art. 16), and also in the horizontal projection of the given line; hence it will be at the intersection of these two

lines. But the point being in the horizontal plane is the same as its horizontal projection (Art. 8); hence the rule:

Produce the vertical projection of the line until it intersects the ground line; at the point of intersection erect a perpendicular to the ground line and produce it until it intersects the horizontal projection of the line; this point of intersection is the required point.

Construction. Produce $m'n'$ to m'; at m' erect the perpendicular $m'm$, and produce it to m. This is the required point.

Second. In the above analysis, by changing the word "vertical" into "horizontal," and the reverse, we have the analysis and rule for finding the point in which the given line pierces the vertical plane.

Construction. Produce mn to o; at o erect the perpendicular oo', and produce it to o'. This is the required point.

Ex. 16. Assume lines similarly situated to KL and MN, Fig. 11, and find where each pierces the H and V planes.

25. Problem 2. **To find the length of a straight line joining two given points in space.**

Let AB, Fig. 18, be the ground line, and (m, m') and (n, n') the two given points.

Analysis. Since the required line contains the two points, its projection must contain the projections of the points (Prop. VI, Art. 13). Hence, if we join the horizontal projections of the points by a straight line, it will be the *horizontal projection* of the line; and if we join the vertical projections of the points, we shall have its *vertical projection.*

If we now revolve the horizontal projecting plane of the line about its horizontal trace until it coincides with the horizontal plane, and find the revolved position of the points, and join them by a straight line, it will be the required distance, since the points do not change their relative position during the revolution.

Construction. Draw mn and $m'n'$. MN will be the required line.

Now revolve its horizontal projecting plane about mn until it

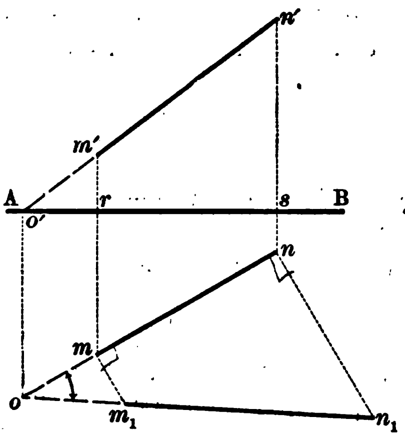


FIG. 18.

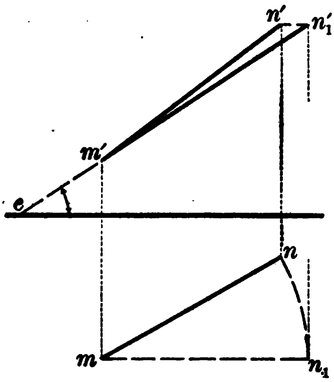


FIG. 19.

coincides with H; the points M and N will fall at m_1 and n_1, at distances from m and n equal to rm' and sn' respectively; join m_1 and n_1, and m_1n_1 will be the required distance.

Since the point o in which the line produced pierces H is in the axis, it remains fixed. The line m_1n_1 produced must then pass through o, and the accuracy of the drawing may thus be verified.

The angle mom_1 is the angle made by the line MN with the horizontal plane. How would you find the angle with the vertical plane?

26. Second method for the same problem.

Analysis. If we revolve the horizontal projecting plane of the line about the projecting perpendicular of either of its points until it becomes parallel to the vertical plane, the line will, in its revolved position, be projected on this plane in its

true length (Prop. XIV, Art 14). If we then construct this vertical projection, it will be the required distance.

Construction. Revolve the horizontal projecting plane, Fig. 19, about the perpendicular at m. The point n describes the arc nn_1 until it comes into the line mn_1 parallel to the ground line; n_1 will be the horizontal projection of N in its revolved position. Its vertical projection must be in $n_1n'_1$ perpendicular to the ground line; and since during the revolution the point N remains at the same distance above H, its vertical projection must also be in the line $n'n'_1$ parallel to the ground line (Prop. I, Art. 7), therefore it will be at n'_1.

The point M, being in the axis, remains fixed, and its vertical projection remains at m'; $m'n'_1$ is then the vertical projection of MN in its revolved position, and the true distance.

By examining the drawing, it will be seen that the true distance is the hypotenuse of a right-angled triangle whose base is the horizontal projection of the line, and altitude the difference between the distances of its two extremities from the horizontal plane. Also, that *the angle at the base is equal to the angle made by the line with the horizontal plane.* Also, that the length of the line is always greater than that of its projection, unless it is parallel to the plane of projection.

Ex. 17. Assume lines situated similarly to KL and MN, Fig. 11, and find the true length of each by the first method. Also determine the angle that each line makes with H and with V.

Ex. 18. Assume a point in the first angle, and one in the third. Find the distance between them by the first method and then by the second. Compare results.

Ex. 19. Find the projections of a point X in MN, Fig. 11, at a distance of 1″ from M.

Ex. 20. Assume a line BC, parallel to the profile plane, and find where it pierces H and V.

Ex. 21. Find a point W in the line MN, Fig. 18, equally distant from H and V.

Ex. 22. A rod $2\frac{1}{2}$ ft. long is suspended horizontally by vertical threads 3 ft. long attached to its extreme ends. How far will the rod be raised by turning it through a horizontal angle of 60°?

Ex. 23. Construct the projections of a $1\frac{1}{2}$ in. cube, one "body diagonal" being perpendicular to H, and the H projection of one edge being perpendicular to the ground line.

Ex. 24. Assume a point P in the third angle. It represents a particle acted upon by three forces not in the same plane. Represent these forces by three lines of different lengths, and find the direction and intensity of their resultant.

27. Problem 3. **To assume a straight line lying in a given plane.**

Every straight line of a plane must pierce any other plane to which it is not parallel in the common intersection of the two. Hence (Prop. XXI, Art. 14) if we take a point in each trace and join the two by a straight line, the line will lie wholly in the plane. Or, we may draw the H projection, and at the points where it intersects the ground line and the horizontal trace erect perpendiculars to the ground line; join the point where the first intersects the vertical trace with the point where the second intersects the ground line — this will be the vertical projection of the line.

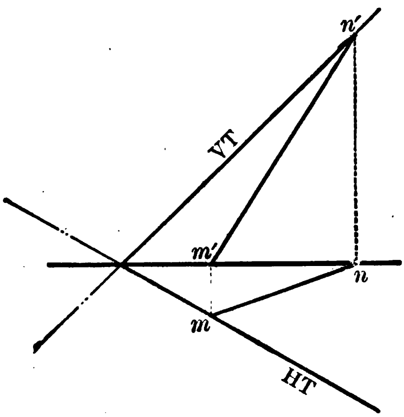


Fig. 20.

Thus in Fig. 20, assume *mn* as the H projection of a line lying in the plane T. The point *m* where it intersects the H trace is vertically projected in the ground line at *m′* (Art. 8).

The point n where it intersects the ground line is vertically projected in the V trace at n' (Art. 8). Joining m' and n', we have the V projection of the line.

Second Case. **If the H projection of the line is parallel to the H trace of the plane in which it lies,** its V projection must be parallel to the ground line (Prop. XXII, Art. 14). Hence we have only to produce the H projection to the ground line, find the V projection of that point, and through it draw the V projection of the line parallel to the ground line. See the line MP, Fig. 21.

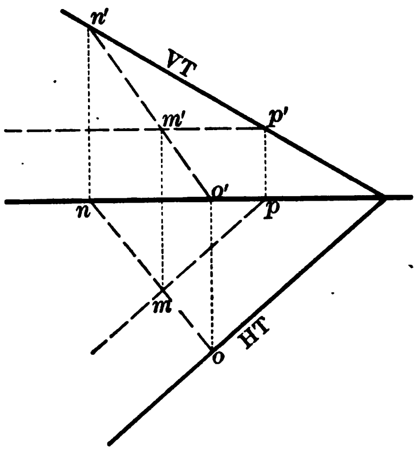


FIG. 21.

Ex. 25. Represent a line CD in the plane R, Fig. 12, assuming the H projection parallel to the ground line.

Ex. 26. Represent a line EF in the plane S, Fig. 12, assuming the V projection parallel to the ground line.

Ex. 27. Represent a line MN in the plane T, Fig. 12, assuming the V projection parallel to the V trace.

28. PROBLEM 4. **Given either projection of a point in an oblique plane, to determine the other projection.**

If through the given projection of the point the corresponding projection of any line of the plane be drawn, and then the other projection of the line be found, the undetermined projection of the point will lie in the corresponding projection of the line (Prop. V, Art. 13), and in a perpendicular to the ground line drawn through the given projection of the point (Prop. XXVI, Art. 16). Thus let m, Fig. 21, be the H projection of a point of the plane T. Through it draw the H projection no

of any line of the plane. This line is vertically projected in $n'o'$ by the previous problem, and the V projection of the point will be found in this line and also in the perpendicular to the ground line mm'. Or, the point m' might have been found by drawing mp, pp', $p'm'$, mm'.

Ex. 28. Represent a point M in each of the planes R, T, and W, Fig. 12, assuming the H projection first and then finding the V projection.

Ex. 29. Represent a point N in each of the planes S, U, and Y, Fig. 12, assuming the V projection first, and then finding the H projection.

29. PROBLEM 5. **To pass a plane through three given points.**

Let M, N, and P, Fig. 22, be the three points.

Analysis. If we join any two of the points by a straight line, it will lie in the required plane, and pierce the planes of projection in the traces of this plane (Prop. XXI, Art. 14). If we join one of these points with the third point, we shall have a second line of the plane. If we find the points in which these lines pierce the planes of projection, we shall have two points of each trace. The traces, and therefore the plane, will be fully determined.

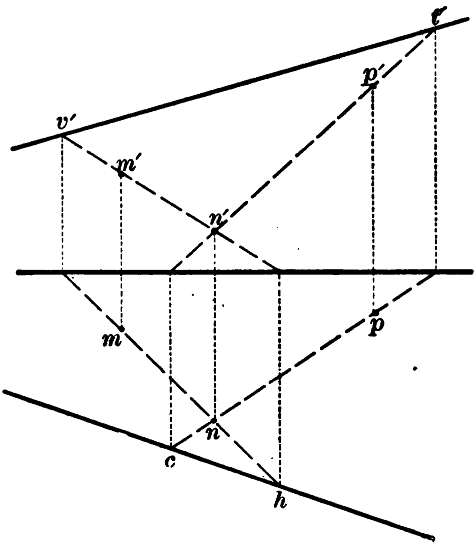


FIG. 22.

Construction. Join m and n by the straight line mn; also m' and n' by $m'n'$. MN will be the line joining the first two points. This pierces H at h, and

V at v', as in Problem 1. Draw also np and $n'p'$; NP will be the second line. It pierces H at c and V at t'. Join h and c by the straight line hc; it is the required horizontal trace. Join v' and t'; $t'v'$ is the vertical trace. Or, produce hc until it meets the ground line, and join this point with either v' or t' for the vertical trace (Prop. VII, Art. 14).

If either MN or NP should be parallel to the ground line, the plane, and consequently its traces, will be parallel to the ground line (Prop. VIII, Art. 14), and it will be necessary to find only one point in each trace.

30. To pass a plane through two straight lines which either intersect or are parallel, we have simply to find the points in which these lines pierce the planes of projection, as in the preceding problem. If the lines do not pierce the planes of projection within the limits of the drawing, then any two points of the lines may be joined by a straight line, and a point in each trace may be determined by finding the points in which this line pierces the planes of projection.

31. To pass a plane through a point and a straight line, join the point with any point of the line by a straight line, and then pass a plane through these lines; or draw through the point a line parallel to the given line, and then pass a plane through the parallels, as above.

Ex. 30. Assume a point A in the second angle, a point B in the third angle, and a point C in the fourth angle, and pass a plane S through them.

Ex. 31. Assume two lines, WX and YZ, each parallel to MN, Fig. 11, and pass through them a plane T.

Ex. 32. Assume two intersecting lines, BC and DE, one parallel to H and oblique to V, the other parallel to V and oblique to H. Pass a plane R through them.

Ex. 33. Assume a line similar to KL, Fig. 11, and a point X in the third angle. Pass a plane T through the point and the line.

Ex. 34. Assume a line MN whose projections do not intersect the ground line within the limits of the drawing. Assume another line OP, parallel to MN, and determine the plane S of the two lines.

Ex. 35. Assume two lines CD and EF each parallel to the ground line, one in the first angle and one in the third angle. Pass a plane T through them.

32. PROBLEM 6. **To pass a plane through a given point parallel to two given straight lines.**

Through the given point draw a line parallel to each of the given lines. The plane of these two lines will be the required plane, since it contains a line parallel to each of the given lines.

33. PROBLEM 7. **To pass a plane through a given straight line parallel to another straight line.**

Through any point of the first line draw a line parallel to the second. Through this auxiliary line and the first pass a plane. It will be the required plane.

Ex. 36. Assume a point C in the second angle, a line MN in the first angle, and a line OP in the third angle. Pass a plane T through the point and parallel to both lines.

Ex. 37. Make the construction for the above problem when MN is parallel to the ground line.

Ex. 38. Assume a line AB in the first angle and a line EF in the fourth angle. Pass a plane S through EF and parallel to AB.

Ex. 39. Make the construction for the above problem when EF is parallel to the ground line.

Ex. 40. Given two lines neither parallel nor intersecting, to pass a plane parallel to both and equally distant from each.

Hint: Draw a straight line from a point in the first line to a point in the second. Through the middle point of this line pass a plane parallel to the two given lines (Art. 32).

34. PROBLEM 8. **Given a line lying in H (or V), and a point in space, to find the position of the point after it has been revolved into H (or V) about the line as an axis.**

Any geometrical magnitude or object is said to be revolved about a straight line as an axis when it is so moved that each of its points describes the circumference of a circle whose plane is perpendicular to the axis, and whose center is in the axis.

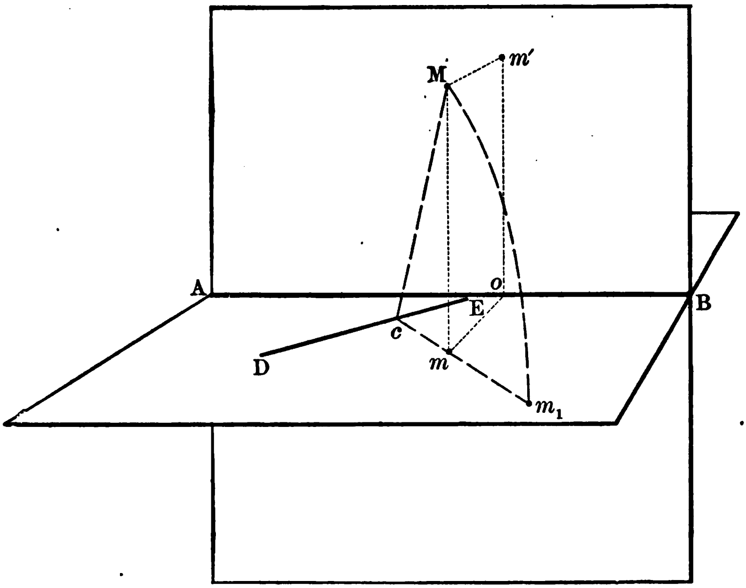


FIG. 23.

If the point M, Figs. 23 and 23 *a*, is revolved about an axis DE, in the horizontal plane, it will describe the circumference of a circle whose center is at *c* and whose radius is M*c*; and since the point must remain in the plane perpendicular to DE, when it reaches the horizontal plane it will be at m_1 in the perpendicular cmm_1, at a distance from *c* equal to M*c*; that is, it will be found *in a straight line passing through its horizontal projection perpendicular to the axis*, and at a distance from the axis

equal *to the hypotenuse of a right-angled triangle of which the base* (mc) *is the distance from the horizontal projection to the axis, and the altitude* (Mm) *equal to the distance of the point from the horizontal plane, or equal to the distance* ($m'o$) *of its vertical projection from the ground line.*

Likewise, if a point be revolved about an axis in the vertical plane until it reaches the vertical plane, its revolved position will be found by the same rule, changing the word *horizontal* into *vertical*, and the reverse.

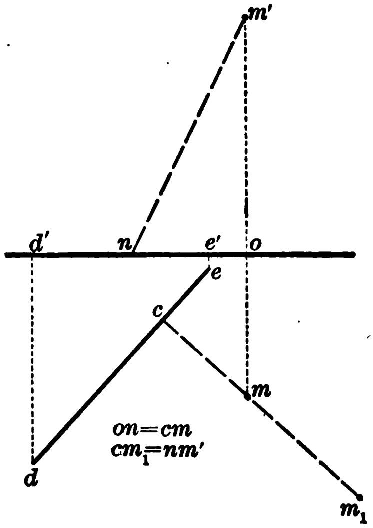


FIG. 23 *a*.

35. PROBLEM 9 *a*. **To find the angle between two straight lines which intersect.**

Let MO and PO, Fig. 24, be the two straight lines, assumed as in Prop. XXVIII, Art. 16.

Analysis. Since the lines intersect, a plane may be passed through them. Revolve this plane about its horizontal trace until it coincides with the horizontal plane, and find the revolved position of the two lines. Since they do not change their relative position, their angle, in this new position, will be the required angle.

Construction. The line MO pierces H at n, and the line PO at p (Art. 24); np is then the horizontal trace of the plane containing the two lines (Art. 30). Revolve this plane about np until it coincides with H. The point O falls at o_1 (Art. 34); the distance so_1 being equal to qo', the hypotenuse of a right triangle whose base bq is equal to the distance so, and whose altitude is the distance of o' from the ground line. The points

n and p, being in the axis, remain fixed; o_1n will then be the revolved position of ON, and o_1p of OP; and the angle no_1p will be the required angle.

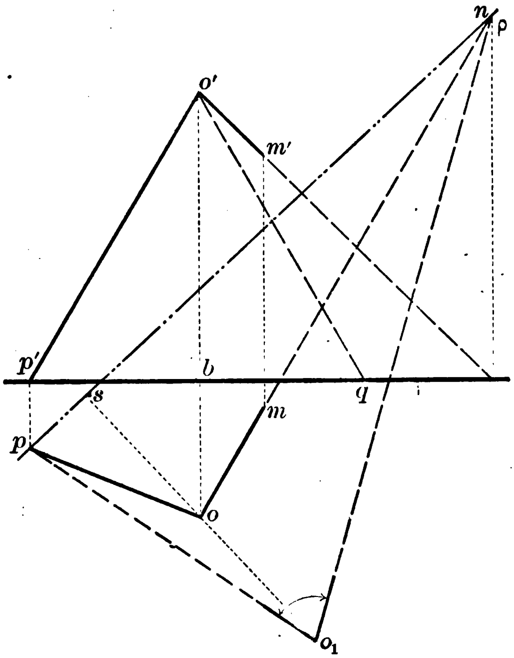


FIG. 24.

Note. The supplement of the angle no_1p may also be regarded as the angle between the lines; but unless otherwise specified the *lesser* of the two will be regarded as the one required.

36. PROBLEM 9*b*. **To find the angle between two intersecting lines, when one of them is parallel to either H or V.**

Let it be required to find the angle between DE and EF, Fig. 25, the latter being parallel to H (Prop. XIV, Art. 14).

In this case it is unnecessary to pass a plane through the two lines, since the line EF may be used as an axis about which ED may be revolved until it is parallel to the H plane. The angle will then be projected upon H in its true magnitude (Prop. XXIII, Art. 14). As a result of this revolution the horizontal projection of any point in the revolved line, as d, will move in a perpendicular to ef a distance from it equal to od_1, the hypotenuse of a right triangle whose base is equal to do and whose altitude is the distance of d' from the V projection of the axis.

During this revolution the vertex E does not move. Hence *ed* takes the new position ed_1, and the angle fed_1 is the angle required.

37. Problem 10. **To find the shortest distance from a given point to a given straight line.**

Analysis. The required distance is the length of a perpendicular from the point to the line. If through the given point and the line we pass a plane, and revolve this plane about either trace until it coincides with the corresponding plane of projection, the line and point will not change their relative positions; hence, if through the revolved position of the point we draw a perpendicular to the revolved position of the line, it will be the required distance.

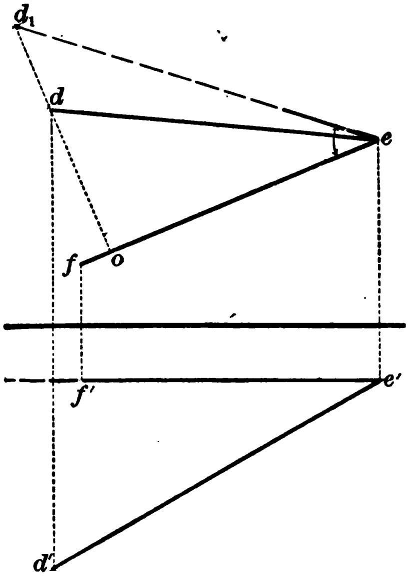


Fig. 25.

Let the student make the construction in accordance with this analysis.

Ex. 41. Assume three points in space, A, B, and C. Join them by lines, forming a triangle. Determine the true size and shape of this triangle by the method of Problems 8 and 9 *a*.

Ex. 42. Assume the projections of a parallelogram in the third angle (Prop. XIX, Art. 14), and determine its true size and shape.

Ex. 43. The H trace of a plane T makes an angle of 60° with the ground line. The angle between the two traces

in space is 75°. What angle does the V trace make with the ground line?

Ex. 44. Assume *a*, *b*, *c*, *d*, *e*, the H projections of the five vertices of a plane pentagon, and a', b', c', the V projections of three of them. Find d', e', and the true figure, without constructing the plane of the pentagon.

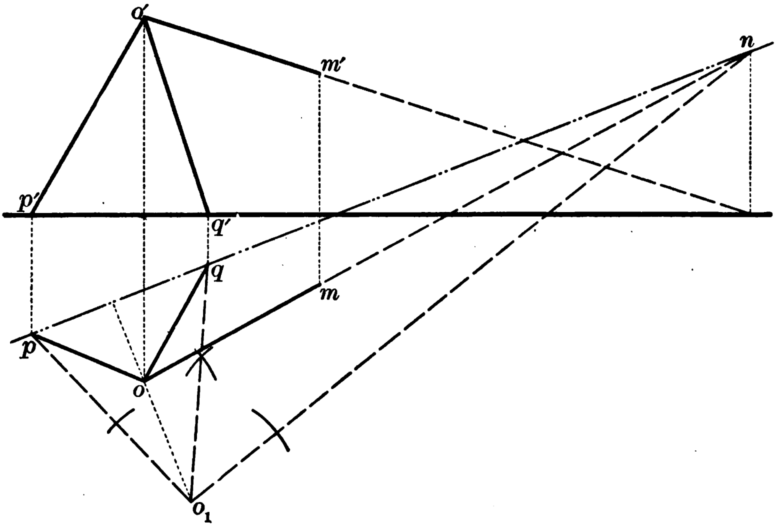


FIG. 26.

38. PROBLEM 11 *a*. **To construct the projections of a line bisecting the angle between two given lines.**

Let OM and OP, Fig. 26, be the two lines, intersecting in the point O.

Analysis. If the plane of the two lines be revolved about its horizontal trace into H, the angle will then be shown in its true size (Art. 35) and may be bisected by a straight line. If the plane be then revolved to its primitive position, and the true position of one point of the bisector be determined, and joined with the vertex of the given angle, we shall have the required line.

Construction. Let po_1n be the revolved position of the angle. Bisect it by o_1q, which will be the horizontal projection of the bisector in its revolved position. Revolving the plane to its primitive position, the point q of the bisector remains fixed, while o_1 returns to o. Joining o and q, we have the H projection of the bisector.

Since the point Q lies in H, its vertical projection must be in the ground line at q'. Joining q' with o', we have the vertical projection of the required bisector.

39. Problem 11 *b.* **To construct the projections of a line bisecting the angle between two given lines, one of which is parallel to H or V.**

In this case the true magnitude of the angle is found as in Fig. 25, the axis of revolution being the given line parallel to one of the planes of projection.

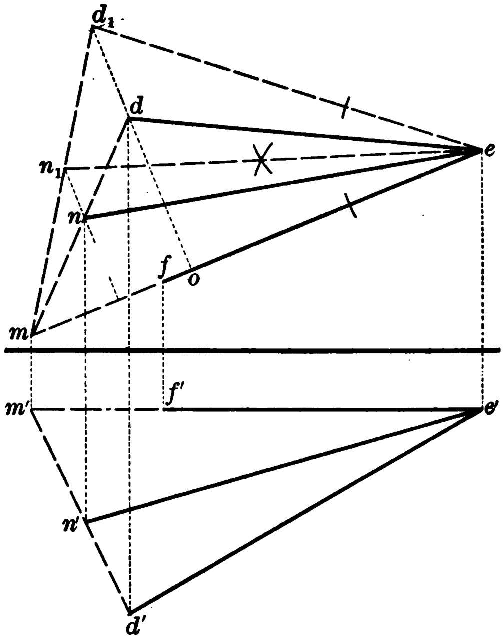


Fig. 27.

Let EF and ED, Fig. 27, be the two given lines, EF being parallel to H. Revolving ED about EF as an axis until it is parallel to H, as in Fig. 25, the true angle between the lines is found to be equal to fed_1. The line en_1 is the H projection of the bisector in its revolved position. To bring it back to

its primitive position, join d_1 with any convenient point in ef, as m. The bisector cuts this auxiliary line in n_1. When ED is revolved to its primitive position, d_1 returns to d, m remains fixed, d_1m takes the position dm, n_1 moves along a perpendicular to the axis (Art. 34) to the position n. en is then the horizontal projection of the bisector. The auxiliary line MD is vertically projected in $m'd'$, N is vertically projected in n' (Prop. V, Art. 13), and $e'n'$ is the vertical projection of the bisector.

Note. The use of an auxiliary line (of which the above is an example) joining the revolved position of a known point with a point in the axis of revolution, is applicable to many problems where it is desired to bring points from their revolved to their primitive positions.

Ex. 45. Assume two intersecting lines, AB and AC, one of which is parallel to the ground line. Find the angle between them, and the projections of the bisector AN.

Ex. 46. Assume two intersecting lines, MN and MP, one of which is perpendicular to V. Find the angle between them and the projections of the bisector.

Ex. 47. Assume two intersecting lines similar to PO and MO, Fig. 26, and determine the projections of the bisector of the supplementary angle formed by producing MO beyond O.

Ex. 48. Assume a line MN and a point P outside the line. Construct the projections of a line PC making an angle of 60° with MN.

Ex. 49. Assume a plane S, and a line AB lying in that plane. Through B pass a line BC lying in the plane and making an angle of 60° with AB.

Ex. 50. Assume a line in the third angle, and a point C outside the line. Construct the projections of a regular hexagon whose center is at the point C and one of whose sides lies in the assumed line.

40. PROBLEM 12 *a*. **To find the intersection of two planes.**

Let T and S, Fig. 28, be the two planes.

Analysis. Since the line of intersection is a straight line contained in each plane, it must pierce the horizontal plane in the horizontal trace of each plane (Prop. XXI, Art. 14); that is, *at the intersection of the two traces.* For the same reason, it must pierce the vertical plane at the intersection of the vertical traces. If these two points be joined by a straight line, it will be the required intersection.

Construction. The required line pierces H at o and V at p': o is its own horizontal projection, and p' is horizontally projected at p; hence po is the horizontal projection of the required line; o is vertically projected at o'; p' is its own vertical projection; and $o'p'$ is the vertical projection of the required line.

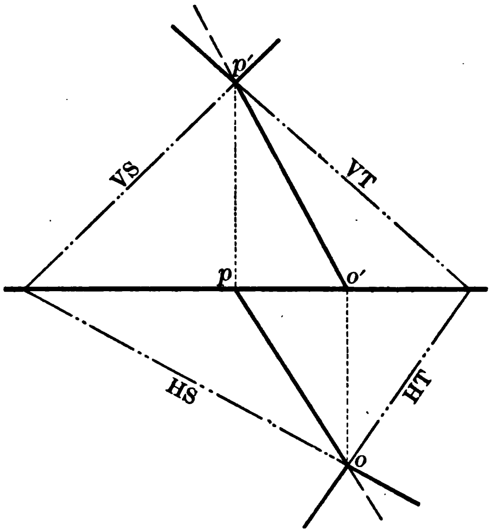


FIG. 28.

41. PROBLEM 12 *b*. **To find the intersection of two planes when either the horizontal or the vertical traces do not intersect within the limits of the drawing.** Let T and S, Fig. 29, be the planes; HT and HS not intersecting within the limits of the drawing.

Analysis. If we pass any plane parallel to the vertical plane, it will intersect each of the given planes in a line parallel to its vertical trace, and these two lines will intersect in a point of

the required intersection. A second point may be determined in the same way, and the straight line joining these two points will be the required line.

Construction. Draw *mn* parallel to the ground line; it will be the H trace of an auxiliary plane. It intersects the two given planes in lines which pierce H at *m* and *n*, and are vertically projected in $m'l'$ and $n'l'$; l' is the vertical, and l the horizontal projection of their intersection. Draw *pq* also parallel to the ground line, and thus determine O. LO is the required line.

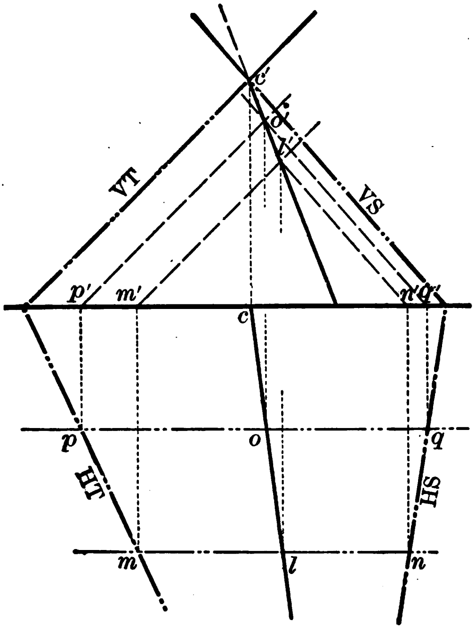


FIG. 29.

Or, since the point C in which the vertical traces intersect is a point of the required line, it may be joined directly to the point L; the second auxiliary plane being unnecessary unless *neither* pair of traces intersect within the limits of the drawing.

Ex. 51. Assume planes similar to R and T, Fig. 12, and find their intersection.

Ex. 52. Assume planes similar to R and S, Fig. 12, but with all four traces intersecting the ground line at the same point. Find the line of intersection by the method of Problem 12 *b*.

Ex. 53. Find the intersection of two planes, both parallel to the ground line. (It will here be convenient to make use of the profile plane of projection.)

Ex. 54. Assume two oblique planes, S and T, and a point M outside of both. Through M pass a line parallel to both planes.

Ex. 55. Given two intersecting planes (oblique) and a point in each, to find the shortest path in these planes between the two points. Analyze and construct.

42. PROBLEM 13 *a*. **To find the point in which a given straight line pierces a given plane.**

Let MN, Fig. 30, be the given line, and T the given plane.

Analysis. If through the line any plane be passed, it will intersect the given plane in a straight line, which must contain the required point. This point must also be on the given line; hence it will be at the intersection of the two lines.

Construction. Let the auxiliary plane be the horizontal projecting plane of the line; *np* is its horizontal and *pp′* its vertical trace (Prop. X, Art. 14). It intersects T in a straight line, which pierces H at *o* and V at *p′*, the vertical projection being *o′p′* (Art. 40). The point *m′*, in which *o′p′* intersects *m′n′*, is the vertical projection of the required point; and *m* is its horizontal projection.

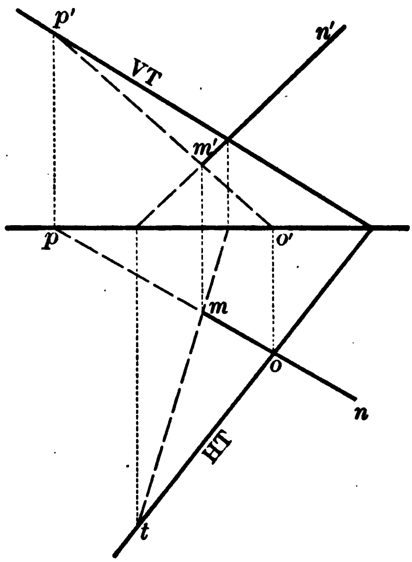


FIG. 30.

The accuracy of the drawing may be verified by using the vertical projecting plane of MN, as an auxiliary plane, and determining *m* directly, as represented in the figure.

Rule. *Produce the* H *projection of the line to the* H *trace and to the ground line. At these points erect perpendiculars to the*

ground line. Draw a straight line from the point where the first intersects the ground line to the point where the second intersects the V *trace. The point where this line cuts the* V *projection of the given line is the* V *projection of the piercing point. This rule will also apply when* H *is changed to* V *and* V *to* H.

Ex. 56. Assume a plane like S, Fig. 12, and a line MN in the second angle, perpendicular to the plane (Prop. XXIV, Art. 14). Find the point C where the line pierces the plane.

Ex. 57. Assume a plane like R, Fig. 12, and a line EB parallel to the ground line. Find the point P where the line pierces the plane.

Ex. 58. Assume a plane T parallel to the ground line, and a line EF parallel to the profile plane. Find where the line pierces the plane.

Ex. 59. Assume a plane like T, Fig. 12, and a point C in the second angle. Find the shortest distance from the point to the plane. See Prop. XXIV (Art. 14), Art. 42, and Art. 25.

Ex. 60. Assume a point P, a line ED, and a plane S. The point P is a source of light. Find the shadow cast by ED upon the plane S.

Ex. 61. Assume a plane like R, Fig. 12, and a line BC in the third angle whose H projection is parallel to HR and whose V projection is parallel to VR. Is this line parallel to the plane? Project the line upon the plane and show the H and V projections of this projection. Remember that the projection of a point upon *any* plane is the foot of a perpendicular dropped from the point to the plane.

Ex. 62. Through a given line MN to pass a plane T perpendicular to a given plane S. Analyze and construct.

Ex. 63. Assume a point P in the third angle, a line MN in the second angle, and a plane T. Pass a plane S through P, parallel to MN and perpendicular to the plane T.

Ex. 64. Assume a plane T and a line MN oblique to it. Pass a line OP lying in the plane and perpendicular to MN. Analyze and construct.

Ex. 65. Assume two lines, AB and CD, not in the same plane, and a point P not lying in either line. Through the point pass a third line PQ touching both AB and CD.

Ex. 66. A ray of light emanating from a given point strikes a given oblique plane. Find the points 1, 2, and 3 where it is successively reflected from the oblique plane and the two planes of projection.

Ex. 67. Assume a plane like R, Fig. 12, and an oblique line EF lying in that plane. Construct the three projections of a cube resting in R, one edge of the lower face being EF.

Ex. 68. Given three oblique lines, AB, CD, EF, no two of which are in the same plane, to construct the projections of a line XZ parallel to AB, and touching CD and EF. Analyze and construct.

43. Problem 13 *b*. **To find the point in which a given straight line pierces a plane, when the plane is given by any two of its straight lines.**

Analysis. If we find the points in which these two lines pierce either projecting plane of the given line, and join these points by a straight line, this will intersect the given line in the required point.

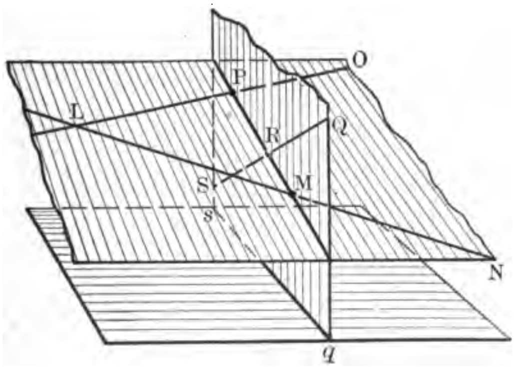


Fig. 31.

Construction. Let LN and LO, Figs. 31 and 31 *a*, be the lines of the given plane, intersecting at L, and QS be the given

line. The line LN pierces the horizontal projecting plane of QS at a point of which m is the horizontal, and m' the vertical projection. LO pierces the same plane at P, and $p'm'$ is the vertical projection of the line joining these two points. This intersects $q's'$ at r', which is the vertical projection of the required point, r being its horizontal projection.

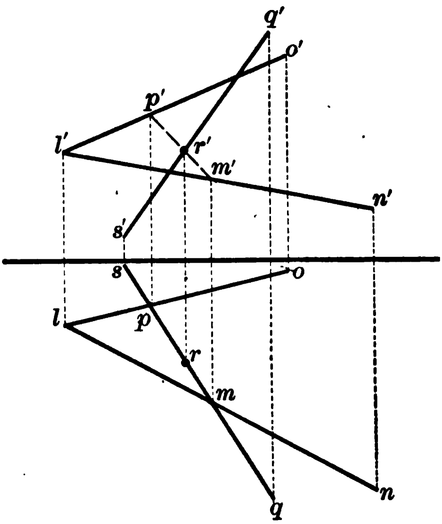


FIG. 31 *a*.

Ex. 69. Assume two parallel lines, AB and CD, in the third angle, and a line XZ. Find the point P where XZ pierces the plane of AB and CD.

Ex. 70. Assume three points in space, A, B, and C, and find the point X in which a line MN pierces the plane of the three points, without constructing the traces of the plane.

44. PROBLEM 14. **Through a given point to pass a plane perpendicular to a given straight line.**

Let M, Fig. 32, be the given point, and NO the given line.

Analysis. Since the plane is to be perpendicular to the line, its traces must be respectively perpendicular to the projections of the line (Prop. XXIV, Art. 14). We thus know the direction of the traces. Through the point draw a line parallel to the horizontal trace; it will be a line of the required plane, and will pierce the vertical plane in a point of the vertical trace. Through this point draw a straight line perpendicular to the vertical projection of the line; it will be the vertical trace of the required plane. Through the point in which this trace inter-

sects the ground line draw a straight line perpendicular to the horizontal projection of the line; it will be the horizontal trace.

Construction. Through m draw mp perpendicular to no; it will be the horizontal projection of a line through M, parallel to the horizontal trace; and since this line is parallel to H, its vertical projection will be $m'p'$ parallel to the ground line. This line pierces V at p' (Art. 24). Draw p'T perpendicular to $n'o'$, and HT perpendicular to no; HT–VT will be the required plane. Or, through M draw a line parallel to the vertical trace. It pierces H in a point of the horizontal trace, and the accuracy of the drawing may thus be tested.

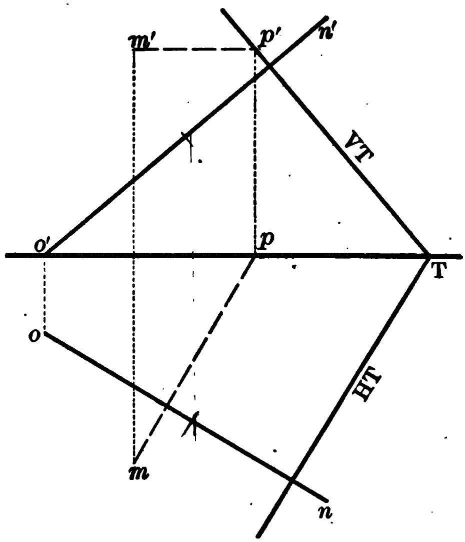


FIG. 32.

Ex. 71. Pass a plane S through a given point M parallel to a given plane T. Analyze and construct.

Ex. 72. Pass a plane S parallel to a given plane R and at a given distance from it. Analyze and construct.

Ex. 73. Given three points, A, B, C, not in the same straight line, to find a point in H equally distant from them. Analyze and construct.

Ex. 74. Assume an oblique plane R and an oblique line MN lying in the plane. Pass a plane T through MN and making an angle of 45° with R. Analyze and construct.

Ex. 75. Assume a plane like R, Fig. 12, and two oblique

lines, AB and CD. Construct the projections of a third line MN, parallel to the plane R and touching AB and CD. Analyze and construct.

Ex. 76. Assume a point M, a line EF, and a plane S. Through M pass a line MX, touching EF and parallel to S. Analyze and construct.

45. PROBLEM 15. **To construct the projections of a circle lying in a given plane, its diameter and the position of its center being known.**

Let the plane T, Fig. 33, be the plane of the circle, and C (assumed as in Art. 28), its center.

Analysis. The projection of any circle upon a plane to which it is neither parallel nor perpendicular will be an *ellipse* whose major axis is the projection of that diameter which is parallel to the trace of the plane of the circle on the plane of projection, and whose minor axis is the projection of the diameter perpendicular to that trace. The length and direction of the major axis being known, it can be drawn at once (Props. XIV and XXII, Art. 14). The extremities of the minor axis may be found by revolving the circle about its trace into the plane of projection, constructing the revolved position of the two axes, and returning them to their primitive positions by the principle noted in Art. 39.

Construction. Revolving the plane T about its horizontal trace into H, the center C falls at c_1, and the circle takes the position $a_1e_1d_1g_1$. The diameter a_1d_1, parallel to HT, is projected upon H in its true length when in its proper position (Prop. XIV, Art. 14). Its H projection is therefore *ad*. The H projection of the diameter e_1g_1 perpendicular to HT, becomes the minor axis of the ellipse. To find its extremities, draw a_1g_1k and e_1d_1m. The points a_1 and g_1 revolve back to *a* and *g* while *k* remains fixed; also points e_1 and d_1 revolve back to *e* and *d*, while *m* remains fixed. The lines then take the

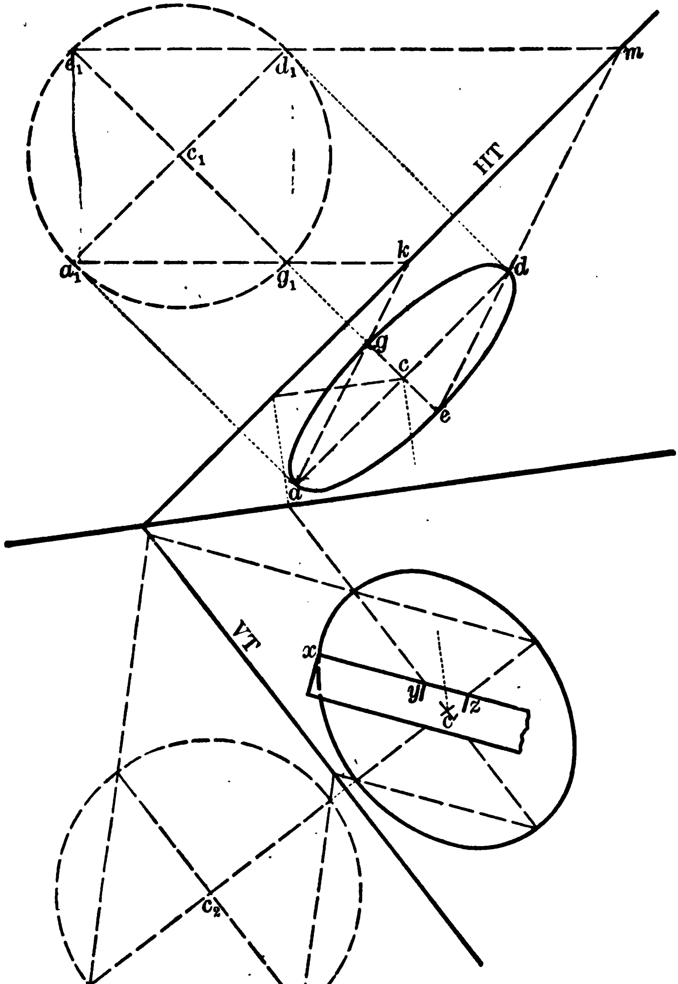


FIG. 33.

positions *kga* and *mde* respectively, and the line *ge* is then the H projection of the diameter GE, and forms the minor axis of the ellipse into which the circle is horizontally projected. The vertical projection may be found by a repetition of this process with respect to the vertical trace, as shown in the figure.

Points on the elliptical curves may be found by the "*trammel method*" thus: Take a stiff strip of paper with a perfectly straight edge *xz* and a sharp corner *x*. Lay off *xy* equal to half the minor axis, and *xz* equal to half the major axis. Then place the strip in various positions, always keeping *y* upon the major and *z* upon the minor axis, and the corner *x* will give points on the ellipse. Having found a sufficient number of them, a smooth curve may be drawn by means of a draftsman's "irregular curve."

Ex. 77. Given a point P and a line MN, to construct the projections of a circle whose center lies in MN and whose circumference passes through P. Analyze and construct.

Ex. 78. Assume a point A in the first angle, a point B in the second angle, and a point C in the third angle. Join them by straight lines forming a triangle ABC. Construct the projections of the inscribed circle.

Ex. 79. Given two intersecting lines and the diameter of a circle tangent to both, to construct the projections of the circle. Analyze and construct.

Ex. 80. A given point P is revolved about a given line MN as an axis through an arc of 90°. Show the projections of its new position and of its path. Analyze and construct.

Ex. 81. Assume a point P, and an oblique line MN in the third angle. The point is revolved about the line as an axis, and pierces H and V in the points X and Y respectively. Locate these points.

Ex. 82. Assume a plane like R, Fig 12, and a point S outside the plane. With S as a vertex construct the three pro-

jections of a right circular cone with base in R and elements making an angle of $22\frac{1}{2}°$ with the axis.

Note. — Exercises 82, 92, and 93 may, if the instructor prefers, be postponed till after the treatment of the generation and classification of surfaces.

46. Problem 16. **To find the angle which a given straight line makes with a given plane.**

Let MN, Fig. 34, be the given line, and T the given plane.

Analysis. The angle made by the line with the plane is the same as that made by the line with its projection on the plane. Hence, if through any point of the line a perpendicular be drawn to the plane, the foot of this perpendicular will be one point of the projection. If this point be joined with the point in which the given line pierces the plane, we shall have the projection of the line on the plane. This projection, the perpendicular, and a portion of the given line form a right-angled triangle, of which the projection is the base, and the angle at the base is the required angle. But the angle at the vertex, that is, *the angle between the perpendicular and the given line*, is the complement of the required angle; hence, if we find the latter angle and subtract it from a right angle, we shall have the required angle.

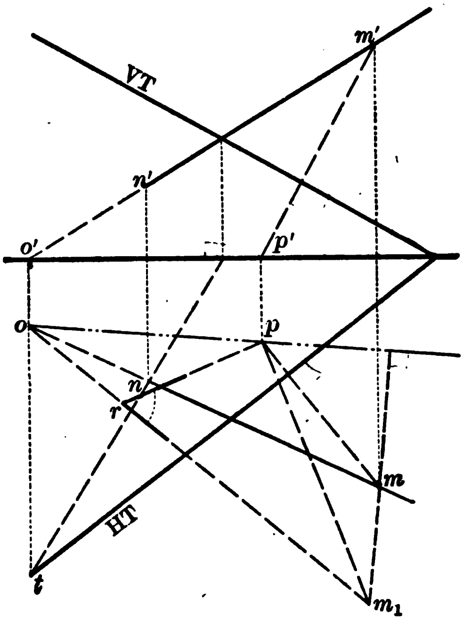


Fig. 34.

Construction. Through M draw the perpendicular MP to the plane T (Prop. XXIV, Art. 14). It pierces H in *p*. The given line pierces H in *o*, and *op* is the horizontal trace of the plane of the two lines (Art. 30). Revolve this plane about *op*, and determine their angle pm_1o, as in Art. 35. Its complement, prm_1, is equal to the required angle.

Ex. 83. Let the above problem be constructed when the plane is parallel to the ground line.

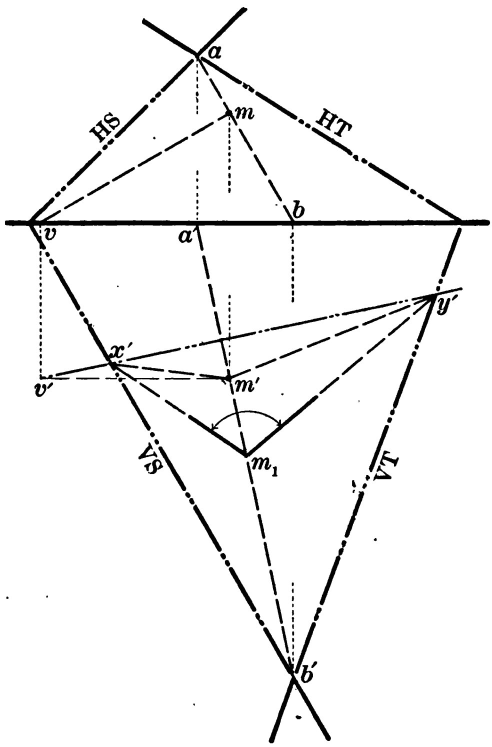


FIG. 35.

47. PROBLEM 17 *a*. **To find the angle between two given planes.**

Analysis. If we pass a plane perpendicular to the intersection of the two planes, it will be perpendicular to both, and cut from each a straight line perpendicular to this intersection at a common point. The angle between these lines will be the measure of the required angle.

Construction. Let the planes S and T, Fig. 35, be the given planes. Their intersection is AB (Art. 40). M is any assumed point in AB. A plane is passed through M perpendicular to AB (Art. 44), and $v'y'$, perpendicular to $a'b'$, is its

vertical trace. This plane cuts the planes S and T in lines, of which $m'x'$ and $m'y'$ are the vertical projections. The angle between the lines MX and MY is found by revolving the auxiliary plane about $x'y'$ into coincidence with V (Art. 35). m_1 is the revolved position of the vertex, and the angle $x'm_1y'$ is the required angle.

48. Second method for the same problem.

Analysis. If from any point in space two lines be passed perpendicular respectively to the two given planes (Prop. XXIV, Art. 14), the angle included between the lines will be equal to the angle between the planes, or its supplement (Art. 35). Let the student make the construction in accordance with this analysis.

49. PROBLEM 17 *b*. **To find the angle between a given plane and either plane of projection,** as the horizontal, we simply pass a plane perpendicular to their intersection (in this case the horizontal trace), as in Fig. 36. This plane cuts on from H, and ON from the plane T, and the angle non_1, found by revolving the auxiliary plane about on (Art. 35), will be the required angle $nn_1 = nn'$.

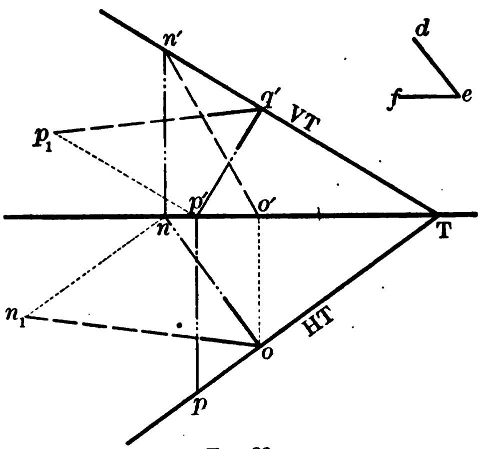


FIG. 36.

In the same way the angle $p'q'p_1$, between the given plane and vertical plane, may be found.

Ex. 84. Find the angle between two planes, both parallel to the ground line.

Ex. 85. Assume two intersecting planes as in Fig. 28, and find the traces of the plane that bisects the dihedral angle between them.

Ex. 86. Given two planes and a straight line, to find a point in the line equally distant from the two planes.

Ex. 87. Assume four points A, B, C, and D, in the third angle. Let them be taken as the vertices of a tetrahedron. Find the angle between two of the faces without drawing the traces of their planes. Analyze and construct.

50. PROBLEM 18. **Either trace of a plane being given, and the angle which the plane makes with the corresponding plane of projection, to construct the other trace.**

Let HT, Fig. 36, be the horizontal trace of the plane, and *def* the angle which the plane makes with the horizontal plane.

Analysis. If a straight line be drawn through any point of the given trace, perpendicular to it, it will be the horizontal trace of a plane perpendicular to the given trace, and if at the same point a line be drawn, making with this line an angle equal to the given angle, this will be the revolved position of a line cut from the required plane by this perpendicular plane (Art. 49). If this line be revolved to its true position, and the point in which it pierces the vertical plane be found, this will be a point of the required vertical trace. If this point be joined with the point where the horizontal trace intersects the ground line, we shall have the vertical trace.

Construction. Through o draw on perpendicular to HT; also on_1, making the angle $non_1 = def$; on_1 will be the revolved position of a line of the required plane. When this line is revolved to its true position, it pierces V at n', nn' being equal to nn_1, and n'T is the required trace.

If the given trace does not intersect the ground line within the limits of the drawing, the same construction may be made at a

second point of the trace, and thus another point of the vertical trace be determined.

Ex. 88. Assume the H trace of a plane T parallel to the ground line. The plane makes an angle of 60° with H. Determine the V trace.

Ex. 89. Assume the H trace of a plane, inclined 30° to the ground line. The plane makes an angle of 45° with V. Determine the V trace. Analyze and construct.

Ex. 90. Given the H trace of a plane T, the projections of a point P, and the distance of the point from the plane, to construct the V trace. Analyze and construct. How many solutions are there?

Ex. 91. Construct the three projections of a 2″ cube, one edge in H making an angle of 30° with the ground line, and one face inclined 30° to H on side *away* from ground line.

Ex. 92. Given a point G, and the H trace of a plane S in which the point lies. Find the V trace, and then construct the projections of a sphere of 2″ diameter tangent to the plane at the point G.

51. PROBLEM 19. **Through a given point to pass a line making given angles with the planes of projection.**

Let it be required to pass a line through C, Fig. 37, making an angle of 30° with H and an angle of 45° with V. If we conceive the line to be of definite length, say 2″, and to be revolved about a vertical line through C until parallel to V, the vertical projection of the line will then be the true length of the line (Art. 26), and its angle with the ground line will be the angle that the line makes with H. Let $(cd_1, c'd'_1)$ be the line in this position. If the line is now revolved back to its original position, the H projection of D will take successive positions in the arc d_1x, and the V projection will take successive positions in the line d'_1d' parallel to the ground line.

Now conceive the line to be revolved about an axis through

C perpendicular to the *vertical* plane until it is parallel to H. The horizontal projection of the line will then be the true length of the line, and its angle with the ground line will be the angle that the line makes with V. $(cd_2, c'd'_2)$ is the line in this position. When the line is revolved back to its original position,

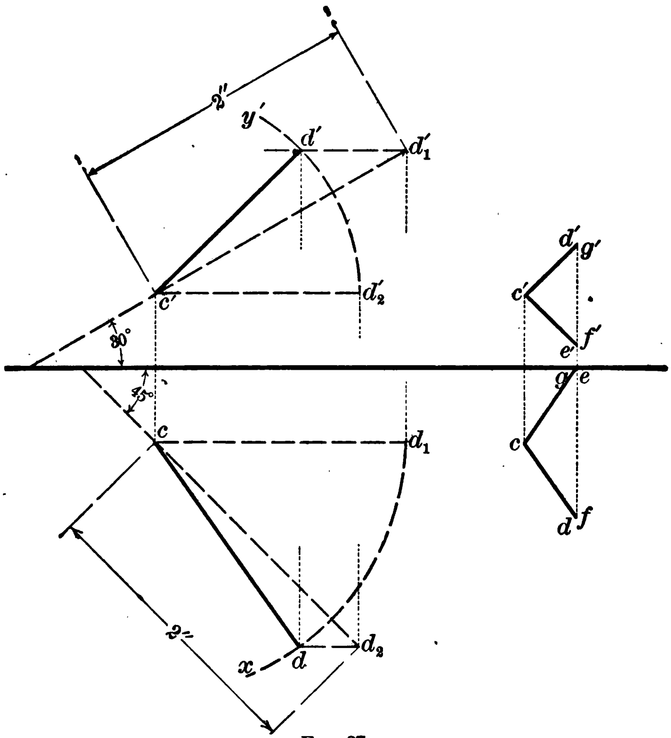


FIG. 37.

the horizontal projection of D will take successive positions in a line through d_2 parallel to the ground line. Where this line intersects the arc d_1x must be the proper position for d. Also the vertical projection of D will, during this revolution, take successive positions in the arc d'_2y', and must therefore be at d', the intersection between this arc and the line d'_1d'.

There are four possible solutions to this problem, shown in Fig. 37 at the right. If a second line be drawn through c' so as to make the same angle with the ground line as the first, and similarly a second line be drawn through c, we may letter these four lines in such a way that they will represent the projections of four different lines, CD, CE, CF, and CG, all making an angle of 30° with H and an angle of 45° with V.

52. PROBLEM 20. **Through a given point to pass a plane making given angles with the planes of projection.**

Analysis. If a line is perpendicular to a plane, the angles between the line and the planes of projection will be the *complements* of the angles between the plane and the planes of projection. Hence if we construct a line making with H and V the complements of the required angles, a plane through the given point perpendicular to this line will make the required angles with H and V. If, for example, it is required to pass a plane making an angle of 40° with H and an angle of 70° with V, we first construct a line making an angle of 90° – 40° with H and an angle of 90° – 70° with V; then we pass through the given point a plane perpendicular to this line.

Note. The sum of the angles made by a *plane* with H and V *cannot be less than* 90°; and the sum of the angles made by a *straight line* with H and V *cannot exceed* 90°.

Ex. 93. Given a point in the third angle as the center of a pulley turning on an inclined shaft making given angles with H and V. Given also the diameter of the pulley, the diameter of the shaft, and the thickness of the pulley, to construct its two projections.

Ex. 94. Construct the three projections of a regular hexagonal prism $3\frac{1}{2}''$ high, each side of whose base is 1″, the base making an angle of 30° with H and 75° with V; one edge of the base being parallel to H; the center, C, being $\frac{3}{4}''$ from H and $1\frac{1}{2}''$ from V.

53. PROBLEM 21. **To find the shortest line that can be drawn, terminating in two straight lines, not in the same plane.**

Let MN, Fig. 38, and OP be the two straight lines.

Analysis. The required line is manifestly a straight line, perpendicular to both of the given lines. If through one of

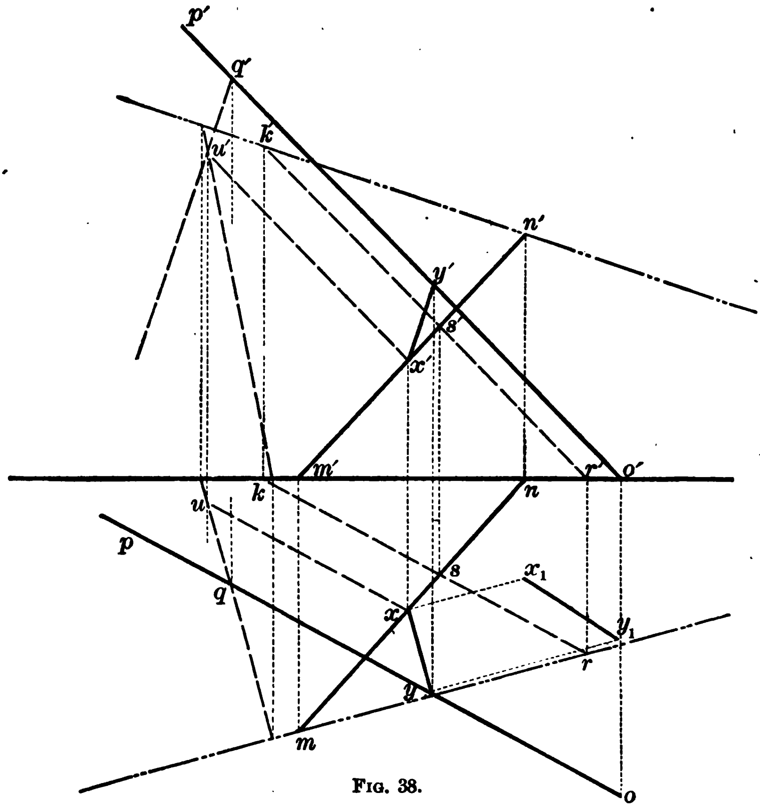


FIG. 38.

the lines we pass a plane parallel to the other, and then project this second line on this plane, this projection will be parallel to the line itself (Prop. XIV, Art. 14), and therefore not parallel to the first line. It will then intersect the first line in a point. If, at this point, we erect a perpendicular to the plane, it will

be contained in the projecting plane of the second line, be perpendicular to both lines, and intersect them both. That portion included between them is the required line.

Construction. Through MN pass a plane parallel to OP (Art. 33). *mr* is its horizontal, and $k'n'$ its vertical trace. Through any point of OP, as Q, draw QU perpendicular to this plane. It pierces the plane at U (Art. 42); and this is one point of the projection of OP on the parallel plane. Through U draw UX parallel to OP; it will be the projection of OP on the plane. It intersects MN in X, which is the point through which the required line is to be drawn, and XY, perpendicular to the plane, is the required line, the true length of which is x_1y_1 (Art. 25).

Ex. 95. Assume two lines, BC and MN, one in the third angle and parallel to the ground line, the other in the second angle and oblique. How can we tell whether they intersect? Find the common perpendicular, XY, to these lines.

Ex. 96. Assume two lines, one of which is perpendicular to H, and find the shortest line between them.

Classification of Lines

54. *Every line may be regarded as generated by the continued motion of a point.* If the generating point be taken in any position on the line, and then be moved to its next position, these two points may be regarded as forming an *infinitely small straight line* or *elementary line.* The two points are *consecutive points*, or points having no distance between them, and may practically be considered as one point.

The line may thus be regarded as made up of an infinite number of infinitely small elements, each element indicating the direction of the motion of the point while generating that part of the line.

55. The law which directs the motion of the generating point determines the nature and class of the line.

If the point moves always in the same direction, that is, so that *the elements of the line are all in the same direction*, the line generated is a *straight line.*

If the point moves so as continually to change its direction from point to point, the line generated is a *curved line*, or *curve.*

If *all the elements of a curve are in the same plane*, the curve is *of single curvature.*

If *no three consecutive elements*, that is, if no four consecutive points, *are in the same plane*, the curve is *of double curvature.*

We thus have **three general classes of lines**:

(1) **Straight** or **right lines**: all of whose points lie in the same direction.

(2) **Curves of single curvature,** or **plane curves**: all of whose points lie in the same plane.

(3) **Curves of double curvature,** or **space curves**: no four consecutive points of which lie in the same plane.

Projection of Curves

56. If all the points of a curve be projected upon the horizontal plane, and these projections be joined by a line, this line is the *horizontal projection* of the curve.

Likewise, if the vertical projections of all the points of a curve be joined by a line, it will be the *vertical projection* of the curve.

57. The two projections of a curve being given, the curve will in general be completely determined. For in the same perpendicular to the ground line two points, one on each projection, may be assumed, and the corresponding point of the curve determined, as in Art. 8. Thus m and m', Fig. 39, being assumed in a perpendicular to the ground line, M will be a point of the

curve, and in the same way every point of the curve may in general be determined.

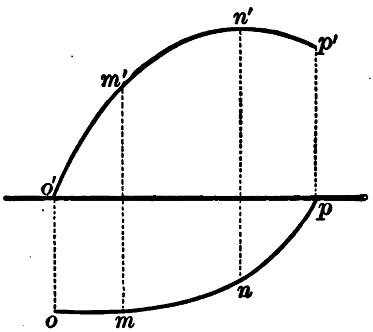


FIG. 39.

58. If the plane of a curve of single curvature is perpendicular to either plane of projection, the projection of the curve on that plane will be a straight line, and all of its points will be projected into the trace of the plane on this plane of projection.

If the plane of the curve be perpendicular to the ground line, both projections will be straight lines, perpendicular to the ground line, and the curve will be undetermined, unless the projections of a sufficient number of its points are determined.

If the plane of the curve be parallel to either plane of projection, its projection on that plane will be equal to itself, since each element of the curve will be projected into an equal element (Prop. XIV, Art. 14). Its projection on the other plane will be a straight line parallel to the ground line.

The projection of a space curve can in no case be a straight line.

59. The points in which a curve pierces either plane of projection can be found by the same rule as in Art. 24. Thus o, Fig. 39, is the point in which the curve MN pierces H, and p' the point in which it pierces V.

TANGENTS AND NORMALS TO LINES

60. Tangent line. If a straight line be drawn through any point of a curve, as M, Fig. 40, intersecting it in another point, as M′, and then the second point be moved along the curve towards M, until it coincides with it, the line containing both

points during the motion will become ***tangent to the curve*** at M, which is the ***point of contact.***

When the point M′ becomes consecutive with M, the line thus containing the element of the curve at M(Art. 54) may, for all practical purposes, be regarded as the tangent; hence we say that ***a straight line is tangent to another line when it contains two consecutive points of that line.***

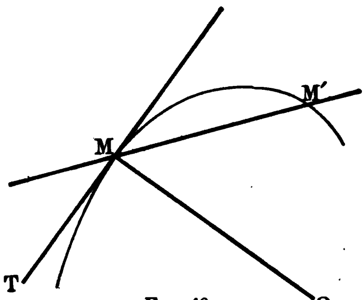


Fig. 40.

Two curves are tangent to each other when they contain two consecutive points, ***or have, at a common point, a common tangent.***

If a straight line is tangent to a plane curve, it will be contained in the plane of the curve. For it passes through two points in that plane, viz. the two consecutive points of the curve.

Also, if a straight line is tangent to another straight line, it will coincide with it, as the two lines have two points in common.

The expression "a tangent to a curve" or "a tangent" will hereafter be understood to mean a rectilinear tangent unless otherwise mentioned.

61. Projections of tangents. ***If two lines are tangent in space, their projections on the same plane will be tangent to each other.*** For the projections of the two consecutive points common to the two lines will also be consecutive points common to the projections of both lines (Art. 56).

The converse of this is not necessarily true. But if both the horizontal and vertical projections are tangent at points which are the projections of a common point of the two lines (Prop. XXVI, Art. 16), the lines will be tangent in space; for the projecting perpendiculars at the common consecutive points will intersect in two consecutive points common to the two lines.

62. Normal. If a straight line be drawn perpendicular to a tangent at its point of contact, as MO, Fig. 40, it is a *normal to the curve.* As an infinite number of perpendiculars can be thus drawn, all in a plane perpendicular to MT at M, there will be an infinite number of normals at the same point.

If the curve be a plane curve, the term "normal" will be understood to mean that normal which is in the plane of the curve unless otherwise mentioned.

63. Rectification. If we conceive a curve to be rolled on its tangent at any point until each of its elements in succession comes into this tangent, the curve is said to be *rectified;* that is, a straight line equal to it in length has been found. Since the tangent to a curve at a point contains the element of the curve, the angle which the curve at this point makes with any line or plane will be the same as that made by the tangent.

64. To rectify a circular arc. Let AB, Fig. 41, be the arc, and AD a tangent to the circle at A. Draw the chord BA, and produce it to E so that AE = $\frac{1}{2}$ BA. With E as center and EB as radius, strike the arc BF. Then AF will be equal to the arc AB, *nearly.* This method is sufficiently accurate only when the angle BCA is less than 60°. For arcs subtending angles greater than 60° the method may be applied to some aliquot part of the arc, and the result spaced off the proper number of times with the dividers. Thus the arc AG is equal to 2 AF or AH.

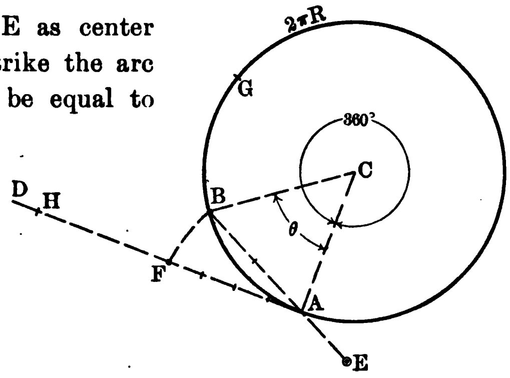


FIG. 41.

If the value of the angle θ is known in degrees, the length of the arc should be *calculated* by means of the relation

$$\frac{\text{Arc AB}}{2\pi R} = \frac{\theta^\circ}{360^\circ},$$

or

$$\text{Arc AB} = 0.01748\, R\theta^\circ.$$

This calculated result may then be laid off with the draftsman's scale.

65. To lay off a straight line upon a circular arc. Let it be required to lay off the distance AF, Fig. 41, upon the circular arc. Divide AF into four equal parts. With the division nearest to A as center, strike an arc through F intersecting the circle in B. The arc AB will then be equal to AF, *nearly*. This method is not accurate when the subtended angle θ is greater than 60°.

66. To lay off any line, curved or straight, upon any other line. Let it be required to lay off upon the line AD, Fig. 42, a length equal to AB; the two curves being tangent at the point A. Separate the points of the dividers a distance B1 such that the chord subtended will nearly equal the arc. Starting at B, step off 1, 2, 3, etc., along the line BA until the point nearest to the point of tangency is reached. Then, without lifting the dividers from the paper, step off from this last point the same number of divisions along the line AD. Then AE will be practically equal to AB. Care should be taken that the points 1, 2, 3, etc., where the dividers rest, are exactly on the line.

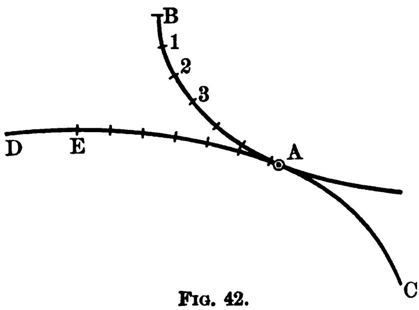


FIG. 42.

Construction of Certain Plane Curves

67. An ellipse may be generated by a point moving in the same plane, so that the sum of its distances from two fixed points shall be constantly equal to a given straight line. The two fixed points are the *foci*.

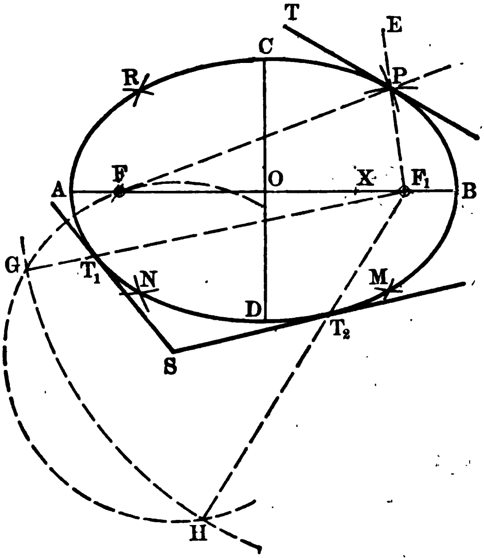


FIG. 43.

To construct an ellipse. Let F and F_1, Fig. 43, be the two foci, and AB the given line, so placed that

$$AF = BF_1.$$

Take any point as X between F and F_1. With F as a center and AX as a radius, describe an arc. With F_1 as a center and BX as a radius, describe a second arc, intersecting the first in the points M and P. These will be points of the required curve, since

$$PF + PF_1 = AX + BX = AB.$$

In the same way all the points may be constructed. A and B are evidently points of the curve, since

$$AF + AF_1 = BF_1 + AF_1 = AB;$$

also $$BF_1 + BF = AB.$$

The point O, midway between the foci, is the *center* of the curve. The line AB, passing through the foci and terminating in the curve, is the *transverse* or *major axis* of the curve.

The points A and B are the *vertices* of the curve. CD, perpendicular to AB at its middle point, is the *conjugate* or *minor axis* of the curve.

If the two axes are given, the foci may be constructed thus: With C, the extremity of the conjugate axis, as a center, and OA, the semi-transverse axis as a radius, describe an arc cutting AB in F and F_1. These points will be the foci, for

$$CF + CF_1 = 2\,OA = AB.$$

To construct a tangent to an ellipse at a point on the curve. Let P be the given point. Draw the focal radii PF and PF_1. The line PT, bisecting the angle FPE, is the required tangent.

To construct a tangent to an ellipse from a point without the curve. Let S be the given point. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points G and H. The lines F_1G and F_1H intersect the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents.

68. A parabola may be generated by a point moving in the same plane, so that its distance from a given point shall be constantly equal to its distance from a given straight line.

The given point is the *focus*, the given straight line the *directrix*.

If through the focus a straight line be drawn perpendicular to the directrix, it is the *axis* of the parabola; and the point in which the axis intersects the curve is the *vertex*.

To construct a parabola. Points on the curve may be constructed from the definition thus: Let F, Fig. 44, be the focus, and EK the directrix. Through F draw FD perpendicular to EK. It will be the *axis*. The point V, midway between F and D, is a point of the curve, and is the *vertex*. Take any point on the axis, as X, and erect the perpendicular XP to the axis. With F as a center, and DX as a radius,

describe an arc cutting XP in the two points P and M. These will be points of the curve, since

$$FP = DX = OP, \text{ also } FM = DX.$$

In the same way all the points may be constructed.

To construct a tangent to a parabola at a point on the curve. Let P be the given point. Draw FP, and the line PO perpendicular to EK. The line PT, bisecting the angle FPO, is the required tangent.

To construct a tangent to a parabola from a point without the curve. Let S be the given point. With S as center and SF as radius, strike an arc cutting the directrix in the points G and H. Lines through these points parallel to the axis DY of the parabola, cut the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents.

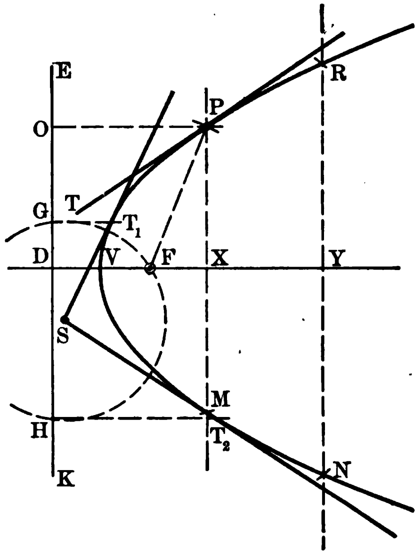


FIG. 44.

69. A hyperbola may be generated by moving a point in the same plane, so that the difference of its distances from two fixed points shall be equal to a given line.

The two fixed points are the *foci*.

To construct a hyperbola. Let F and F_1, Fig. 45, be the two foci, and AB the given line. Select any point, as X, on the line AB produced, and with F as a center and AX as a radius, strike an arc. Then with F_1 as center and BX as

radius, strike an arc cutting the first arc in the points P and M. These will be points on the curve, since $AX - BX = AB$; and hence $FP - F_1P = AB$, and $FM - F_1M = AB$. If F_1 is taken as center and AX as radius, and then F as center with BX as radius, we shall get points R and N on the other branch

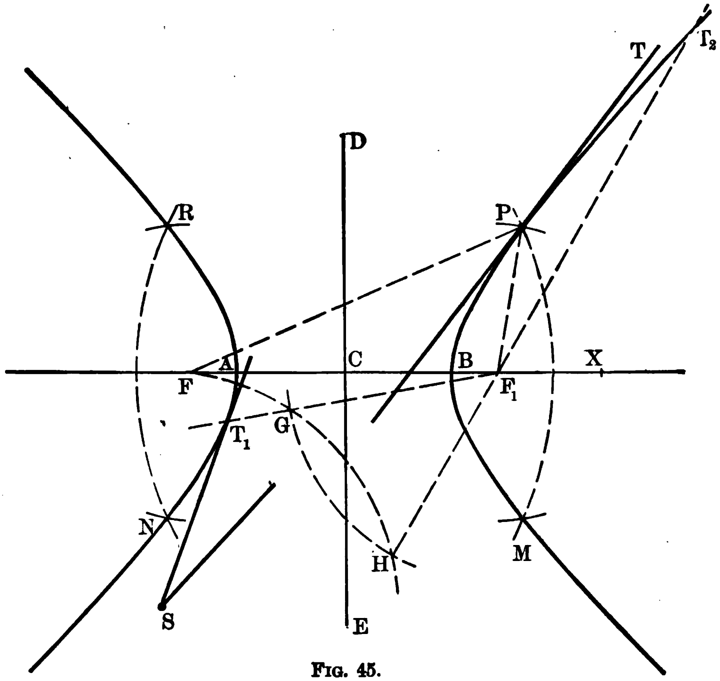


FIG. 45.

of the hyperbola. By selecting other points than X on the line AB produced, and proceeding in a similar manner, any number of points on the curve may be determined.

The points A and B are the *vertices* of the hyperbola. The point C, midway between the foci, is the *center*, and AB is the *transverse axis*. A perpendicular, DE, to the transverse axis at the center is the *indefinite conjugate axis*. It evidently does not intersect the curve.

To construct a tangent to the hyperbola at a point on the curve. Let P be the given point. Draw the focal radii FP and F_1P. The line PT, bisecting the angle FPF_1, is the required tangent.

To construct a tangent to the hyperbola from a point without the curve. Let S be the given point. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points G and H. Then F_1G and F_1H intersect the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the tangents.

If tangents are passed to the hyperbola through the center C, the points of contact will be at infinity, and the lines are called *asymptotes*.

The Helix

70. If a point be moved uniformly around a straight line, remaining always at the same distance from it, and having at the same time a uniform motion in the direction of the line, it will generate a curve of double curvature, called a ***helix***.

The straight line is the *axis* of the curve.

71. To construct the projections of the helix. Since all the points of the curve are equally distant from the axis, the projection of the curve on a plane perpendicular to this axis will be the circumference of a circle.

Thus, let *m*, Fig. 46, be the horizontal, and $m'n'$ the vertical projection of the axis, and P the generating point, and suppose that while the point moves once around the axis, it moves through the vertical distance $m'n'$ (called the *pitch* of the helix); *prqs* will be the horizontal projection of the curve.

To determine the vertical projection, divide *prqs* into any number of equal parts, as sixteen, and also the line $m'n'$ into the same number, as in the figure. Through these points of division draw lines parallel to the ground line. Since the motion

of the point is uniform, while it moves one eighth of the way round the axis it will ascend one eighth of the distance $m'n'$, and be horizontally projected at x, and vertically at x'. When the point is horizontally projected at r, it will be vertically projected at r'; and in the same way the points y', q', etc., may be determined, and $p'r'q's'$ will be the required vertical projection.

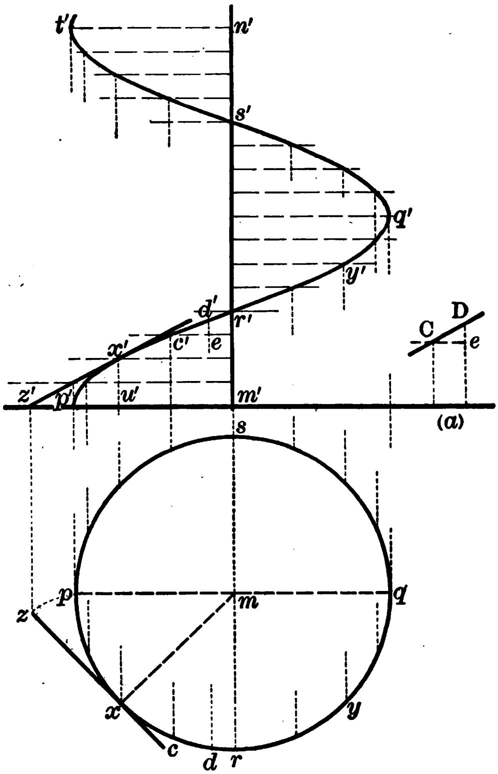


FIG. 46.

72. To determine the slope of the helix. It is evident from the nature of the motion of the generating point, that in generating any two equal portions of the curve, it ascends the same vertical distance; that is, any two elementary arcs of the curve will make equal angles with the horizontal plane. Thus, if CD (a), Fig. 46, be any element of the curve, the angle which it makes with the horizontal plane will be DCe, or the angle at the base of a right-angled triangle of which Ce=cd, Fig. 46, is the base, and De the altitude. But from the nature of the motion, Ce is to De as any arc px is to the corresponding ascent $u'x'$. Hence, if

we rectify the arc xp (Art. 63), and with this as a base construct a right-angled triangle, having $x'u'$ for its altitude, the angle at the base will be the angle which the arc, or its tangent at any point, makes with the horizontal plane. Therefore,

73. To construct the projections of a tangent to the helix at a given point, as X, we draw xz, Figs. 46 and 46 *a*, tangent to the circle pxr at x; it will be the horizontal projection of the required tangent. On this, from x, lay off the rectified arc xp to z (Art. 64); z will be the point where the tangent pierces H, and $z'x'$ will be its vertical projection.

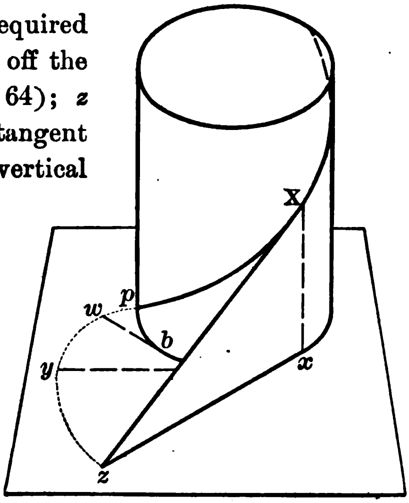


FIG. 46 *a*. Development of Helix.

$$Xp = Xz = \sqrt{\overline{Xx}^2 + \overline{xz}^2} = \sqrt{\overline{Xx}^2 + \overline{xbp}^2}$$

74. To find the length of any given portion of the helix. Since the angle which a tangent to the helix makes with the horizontal plane is constant, and since each element of the curve is equal to the hypotenuse of a right-angled triangle of which the base is its horizontal projection, the angle at the base is the constant angle, and the altitude is the ascent of the point while generating the element, it follows that when the helix is rolled out on its tangent, the sum of the elements, or length of any portion of the curve, will be equal to the hypotenuse of a right-angled triangle, of which the base is its horizontal projection rectified, and the altitude is the ascent of the generating point while generating the portion considered. Thus the length of the arc pX, Figs. 46 and 46 *a*, is equal to the length of the portion of the tangent ZX.

CLASSIFICATION OF LINES AND SURFACES

- **Lines**
  - Straight lines
  - Plane curves (circle, ellipse, parabola, hyperbola, spirals, roulettes, etc.)
  - Space curves (helix, spherical epicycloid, loxodrome, etc.)
- **Surfaces**
  - **Ruled Surfaces** (Generated by Straight Lines)
    - **Planes**
    - **Single-curved Surfaces** (Developable)
      - All positions of generatrix parallel (cylinders)
      - All positions of generatrix intersect in common point (cones)
      - Consecutive positions of generatrix intersect two and two; no three positions intersecting in common point; may be generated by a system of tangents to line of double curvature (convolutes)
    - **Warped Surfaces** No two consecutive positions of generatrix in same plane; hence, non-developable.
      - Plane directer and two linear directrices
        - Directrices both straight (hyperbolic paraboloid)
        - One straight and one curved directrix (conoid)
        - Directrices both curved (cylindroid)
      - Three linear directrices
        - Directrices all straight (hyperboloid of one nappe)
        - One curve, two straight line directrices
        - Two curves, one straight line directrix (corne de vache)
        - Directrices all curved
      - Surfaces generated by a straight line touching a space curve and another straight line, and making a constant angle with a fixed plane (helicoid)
  - **Double-curved Surfaces** (Generated by Curved Lines)
    - **Surfaces of Revolution** (sphere, ellipsoid of revolution, torus, paraboloid of revolution, etc.)
    - Surfaces of double curvature not generated by a revolving line nor by a straight line (ellipsoid; elliptical paraboloid; elliptical hyperboloid)

Generation and Classification of Surfaces

75. Generation of surfaces. A surface *may be generated by the continued motion of a line.* The moving line is the *generatrix* of the surface; and the different positions of the generatrix are the *elements.*

If the generatrix be taken in any position, and then be moved to its next position on the surface, these two positions are *consecutive positions* of the generatrix, or *consecutive elements* of the surface, and may practically be regarded as one element.

76. Two great classes of surfaces. The form of the generatrix, and the law which directs its motion, determine the nature and class of the surface.

Surfaces may be divided into *two* general classes.

First. *Those which can be generated by straight lines, or which have rectilinear elements.* These are **ruled surfaces.**

Second. *Those which can be generated only by curves, and which can have no rectilinear elements.* These are **double-curved surfaces.**

77. There are **three classes of ruled surfaces**:

(1) **Planes**: which may be generated by *a straight line moving so as to touch another straight line, and having all its positions parallel to its first position.*

(2) **Single-curved surfaces,** *or developable surfaces*: which may be generated by *a straight line moving so that any two of its consecutive positions shall be in the same plane.*

(3) **Warped surfaces**: which may be generated by *a straight line moving so that no two of its consecutive positions shall be in the same plane.*

78. There are **three classes of single-curved surfaces:**

(1) **Cylinders**: those in which *all the positions* of the rectilinear generatrix are parallel.

(2) **Cones**: those in which *all the positions* of the rectilinear generatrix intersect in a common point.

(3) **Convolutes**: those in which the consecutive positions

of the rectilinear generatrix intersect *two and two*, no three positions intersecting in a common point.

CYLINDRICAL SURFACES, OR CYLINDERS

79. Cylinders in general. Single-curved surfaces of the first kind are *cylindrical surfaces*, or *cylinders*. Every cylinder may be generated by moving *a straight line so as to touch a curve, and have all its positions parallel.*

The moving line is the *rectilinear generatrix.* The curve is the *directrix.* The different positions of the generatrix are the *rectilinear elements of the surface.*

Thus, in Fig. 47, if the straight line MN be moved along the curve *mlo*, having all its positions parallel to its first position, it will generate a cylinder.

If the cylinder be intersected by any plane not parallel to the rectilinear elements, the curve of intersection may be taken as a *directrix*, and any rectilinear element as the generatrix, and the surface be regenerated. This curve of intersection may also be the *base of the cylinder.*

The intersection of the cylinder by one of the planes of projection, as the horizontal, is usually taken as the base. If this base have a center, the straight line through it, parallel to the rectilinear elements, is the *axis of the cylinder.*

A definite portion of the surface included by two parallel planes is sometimes considered; in which case the lower curve of intersection is the *lower base*, and the other the *upper base.*

Cylinders are distinguished by the name of their bases; as a cylinder with a circular base, a cylinder with an elliptical base, etc.

If the rectilinear elements are perpendicular to the plane of the base, the cylinder is a *right cylinder*, and the base a *right section.*

A cylinder may also be generated by moving the curvilinear directrix, as a generatrix, along any one of the rectilinear ele-

ments, as a directrix, the curve remaining always parallel to its first position.

If the curvilinear directrix be changed to a straight line, the cylinder becomes a plane.

It is manifest that if a plane parallel to the rectilinear elements intersects the cylinder, the lines of intersection will be rectilinear elements which will intersect the base.

80. Projecting cylinders. It will be seen that the projecting lines of the different points of a curve (Art. 56) form a right cylinder, the base of which, in the plane of projection, is the projection of the curve.

These cylinders are respectively the *horizontal* and *vertical projecting cylinders* of the curve, and by their intersection determine the curve.

81. To construct the projections of a cylinder. A cylinder is represented by projecting one or more of the curves of its surface and its principal rectilinear elements.

When these elements are not parallel to the horizontal plane, it is usually represented thus: Draw the base as *mlo*, Fig. 47, in the horizontal plane. Tangent to this draw straight lines *lx* and *kd* parallel to the horizontal projection of the generatrix; these will be the horizontal projections of the extreme rectilinear elements as seen from the point of sight, thus forming the horizontal projection of the cylinder. Draw tangents to the base perpendicular to the ground line as *mm'*, *oo'*; through the points *m'* and *o'* draw lines *m'n'* and *o's'* parallel to the vertical projection of the generatrix, thus forming the vertical projection of the cylinder, *m'o'* being the vertical projection of the base.

82. To assume a rectilinear element, we have simply to draw a line parallel to the rectilinear generatrix through any point of the base, as U, Fig. 47. Then *ub* and *u'b'* will be the projections of the element.

To assume a point of the surface. Let p, Fig. 47, be the horizontal projection of the point. Through p draw pq parallel to lx; it will be the horizontal projection of two elements, one of which pierces H at q and the other at u; and $q'y'$ and $u'b'$

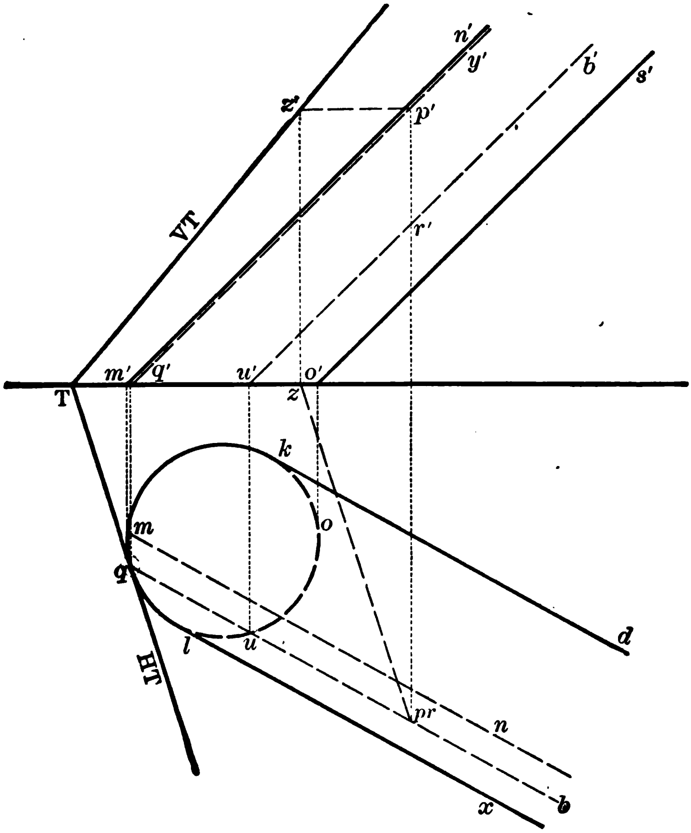


FIG. 47.

will be the vertical projections of these elements, one on the upper and one on the lower side of the cylinder; and p' and r' are the vertical projections of the two points of the surface horizontally projected at pr.

CONICAL SURFACES, OR CONES

83. Single-curved surfaces of the second kind are *conical surfaces*, or *cones*.

Every cone may be generated by moving a straight line so as continually to touch a given curve and pass through a given point not in the plane of the curve.

The moving line is the *rectilinear generatrix;* the curve, the *directrix;* the given point, the *vertex* of the cone; and the different positions of the generatrix, the *rectilinear elements*.

The generatrix being indefinite in length will generate two parts of the surface on different sides of the vertex which are called *nappes;* one the *upper*, the other the *lower*, *nappe*.

Thus, if the straight line MS, Fig. 48, move along the curve *mlo* and continually pass through S, it will generate a cone.

If the cone be intersected by any plane not passing through the vertex, the curve of intersection may be taken as a *directrix* and any rectilinear element as a *generatrix*, and the cone be regenerated. This curve of intersection may also be the *base* of the cone. The intersection of the cone by the horizontal plane is usually taken as the base.

If a definite portion of either nappe of the cone included between two parallel planes is considered, it is called a *frustum of the cone;* one of the limiting curves being the *lower* and the other the *upper base* of the frustum.

Cones are distinguished by the names of their bases; as a cone with a circular base, a cone with a parabolic base, etc.

If the rectilinear elements all make the same angle with a straight line passing through the vertex, the cone is a *right cone*, the straight line being its *axis*.

If the curvilinear directrix of a cone be changed to a straight line, or if the vertex be taken in the plane of the curve, the cone will become a plane.

If the vertex be removed to an infinite distance, the cone will evidently become a cylinder.

If a cone be intersected by a plane through the vertex, the lines of intersection will be rectilinear elements intersecting the base.

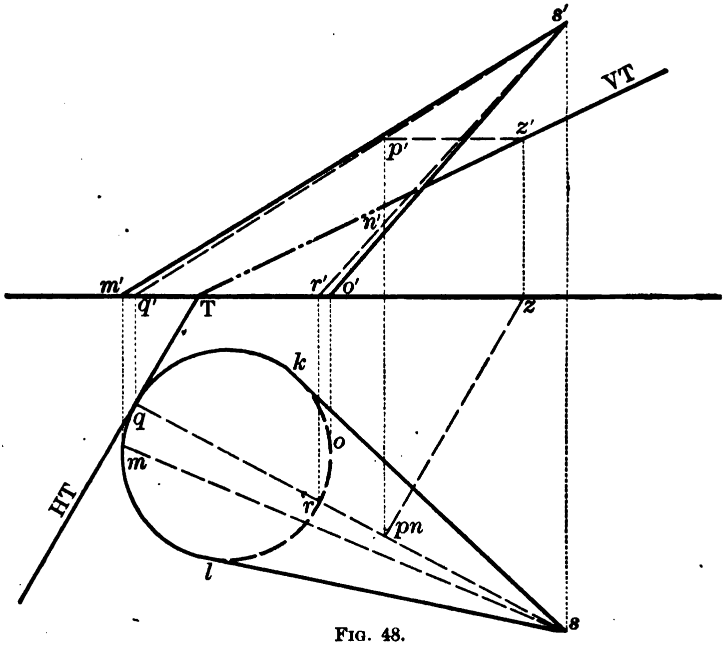


FIG. 48.

84. To construct the projections of a cone. A cone is represented by projecting the vertex, one of the curves on its surface, and its principal rectilinear elements. Thus let S, Fig. 48, be the vertex. Draw the base *mlo* in the horizontal plane, and tangents to this base through *s*, as *sl* and *sk*, thus forming the horizontal projection of the cone. Draw tangents to the base perpendicular to the ground line, as *mm'*, *oo'*, and through *m'* and *o'* draw the straight lines *m's'* and *o's'*, thus forming the vertical projection of the cone.

85. To assume a rectilinear element, we have simply to draw through any point Q of the base a straight line to the vertex. Thus qs and $q's'$ represent an element of the cone through Q.

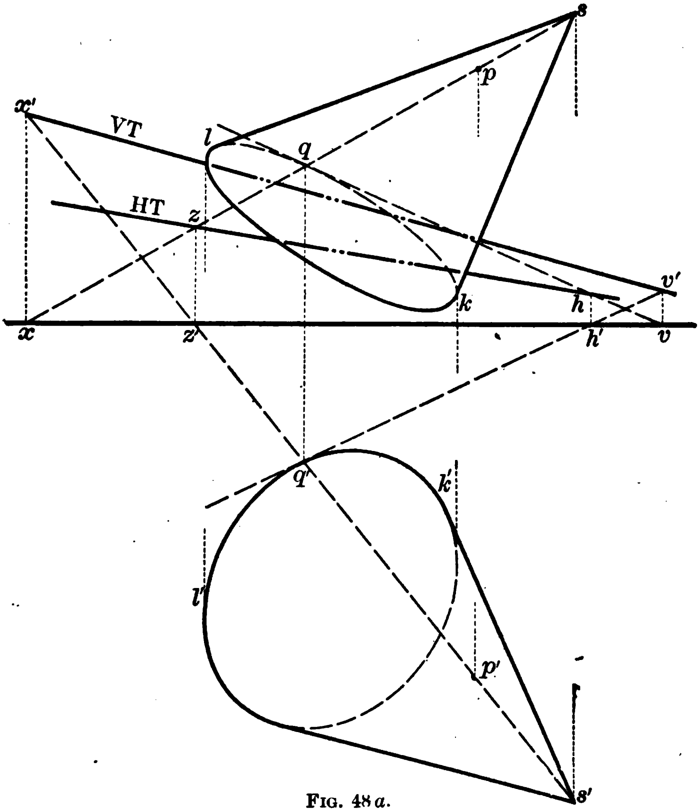


FIG. 48 *a*.

To assume a point on the surface, we first assume one of its projections, as the horizontal, pass an element through it, and find the vertical projection of the point on the vertical projection of the element, as in Art. 82.

Note. While a cylinder or cone is usually represented with the base in one of the planes of projection, it is sometimes more

convenient to assume the base in other positions. Thus in Fig. 48 *a*, the base may be any plane curve or space curve. An element SQ or point P of the surface would be represented as shown.

PLANES TANGENT TO SURFACES

86. A plane is tangent to a surface when it has at least one point in common with the surface, through which, if any intersecting plane be passed, the straight line cut from the plane will be tangent to the line cut from the surface at the point. This point is the *point of contact.*

It follows from this definition that the tangent plane is *the locus of, or place in which are to be found, all straight lines tangent to lines of the surface at the point of contact;* and since any two of these straight lines are sufficient to determine the tangent plane, we have the following general rule for passing a plane tangent to any surface at a given point: **Draw any two lines of the surface intersecting at the point. Tangent to each of these, at the same point, draw a straight line. The plane of these two tangents will be the required plane.**

87. A straight line is normal to a surface at a point, when *it is perpendicular to the tangent plane at that point.*

A plane is *normal to a surface* when *it is perpendicular to the tangent plane at the point of contact.* Since any plane passed through a normal line will be perpendicular to the tangent plane, there may be an infinite number of normal planes to a surface at a point, while there can be but one normal line.

To pass a line normal to a surface at a given point, we first determine the traces of the tangent plane at that point, and then pass a line through the point perpendicular to the plane.

88. PROPOSITION XXXI. **The plane tangent at any point of a surface with rectilinear elements must contain the rectilinear**

elements that pass through the point of contact. For the tangent to each rectilinear element is the element itself (Art. 60), and this tangent must lie in the tangent plane (Art. 86).

89. PROPOSITION XXXII. **A plane which contains a rectilinear element of a single-curved surface, and its consecutive one, will be tangent to the surface at every point of this element, or all along the element.** For if through *any point* of the element *any intersecting plane* be passed, it will intersect the consecutive element in a point consecutive with the first point. The straight line joining these two points will lie in the given plane and be tangent to the line cut from the surface (Art. 60). Hence the given plane will be tangent at the assumed point.

This element is the *element of contact.*

90. PROPOSITION XXXIII. *Conversely*, **If a plane be tangent to a single-curved surface, it must, in general, contain two consecutive rectilinear elements.** For if through any point of the element contained in the tangent plane (Art. 88) we pass an intersecting plane, it will cut from the surface a line, and from the plane a straight line, which will have two consecutive points in common (Art. 86). Through the point consecutive with the assumed point draw the element consecutive to the first element; it must lie in the plane of the second point and first element (Art. 75); that is, in the tangent plane.

91. It follows from these principles, that if a plane be tangent to a single-curved surface, and the element of contact be intersected by any other plane, the straight line cut from the tangent plane will be tangent to the line cut from the surface. Hence, if the base of the surface be in the horizontal plane, *the horizontal trace of the tangent plane must be tangent to this base*, at the point in which the element of contact pierces the horizontal plane; and the same principle is true if the base lie in the vertical plane.

Problems relating to Planes Tangent to Cylinders and Cones

92. The solution of the following problems depends mainly upon the principles that a plane tangent to a single-curved surface is tangent all along a rectilinear element (Art. 89); and that if such surface and tangent plane be intersected by any plane, the lines of intersection will be tangent to each other (Art. 86).

93. Problem 22. **To pass a plane tangent to a cylinder at a given point on the surface.**

Let the cylinder be given as in Art. 81, Fig. 47, and let P be the point, assumed as in Art. 82.

Analysis. Since the required plane must contain the rectilinear element through the given point, and its horizontal trace must be tangent to the base at the point where this element pierces the horizontal plane (Art. 91), we draw the element, and at the point where it intersects the base, a tangent; this will be the horizontal trace. The vertical trace must contain the point where this element pierces the vertical plane, and also the point where the horizontal trace intersects the ground line. A straight line joining these two points will be the vertical trace. When this element does not pierce the vertical plane within the limits of the drawing, we draw through any one of its points a line parallel to the horizontal trace; it will be a line of the required plane, and pierce the vertical plane in a point of the vertical trace.

Construction. Draw the element PQ; it pierces H at q. At this point draw qT tangent to mlo; it is the required horizontal trace. Through P draw PZ parallel to qT (Art. 27); it pierces V at z', and z'T is the vertical trace.

$mm'n'$ is the plane tangent to the surface along the element MN.

94. PROBLEM 23. **To pass a plane through a given point without the surface tangent to a cylinder.**

Let the cylinder be given as in the preceding problem.

Analysis. Since the plane must contain a rectilinear element, if we draw a line through the given point, parallel to the rectilinear elements of the cylinder, it must lie in the tangent plane, and the point in which it pierces the horizontal plane will be one point of the horizontal trace. If through this point we draw a tangent to the base, it will be the required horizontal trace. A line through the point of contact, parallel to the rectilinear elements, will be the element of contact, and the vertical trace may be constructed as in the preceding problem. Or a point of this trace may be obtained by finding the point in which the auxiliary line pierces the vertical plane.

Two or more tangent planes may be passed if two or more tangents can be drawn to the base from the point in which the auxiliary line pierces the horizontal plane.

Let the construction be made in accordance with the above analysis.

95. PROBLEM 24. **To pass a plane tangent to a cylinder and parallel to a given straight line.**

Let the cylinder be given as in Fig. 49, and let RS be the given line.

Analysis. Since the required plane must be parallel to the rectilinear elements of the cylinder as well as to the given line, if a plane be passed through this line parallel to a rectilinear element, it will be parallel to the required plane, and its traces parallel to the required traces (Prop. XII, Art. 14). Hence a tangent to the base, parallel to the horizontal trace of this auxiliary plane, will be the required horizontal trace. The element of contact and vertical trace may be found as in the preceding problem.

Construction. Through RS pass the plane F, parallel to MN. Tangent to ***mlo*** and parallel to HF, draw HT; it is the horizontal trace of the required plane, and VT, parallel to VF, is

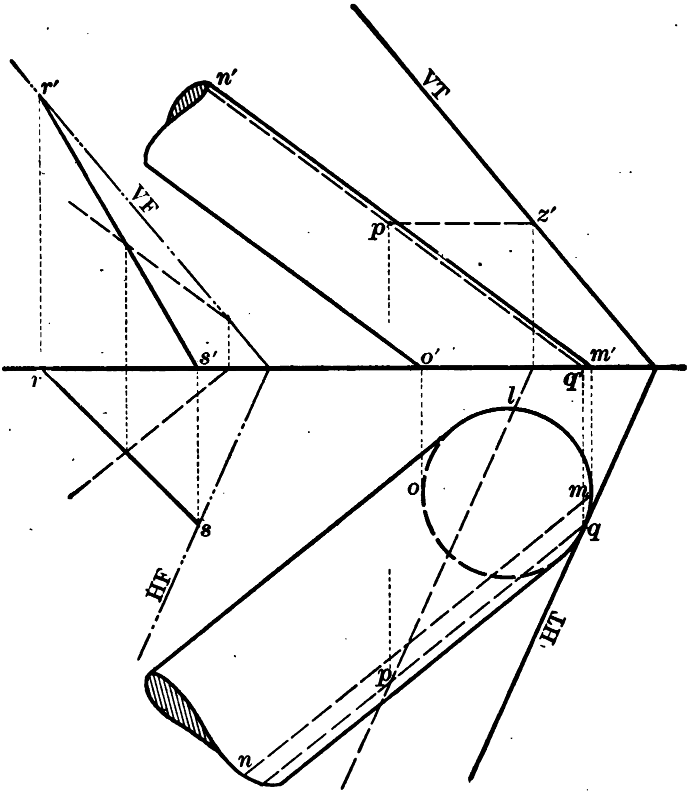


Fig. 49.

the vertical trace. QP is the element of contact. The point z', determined as in the preceding problem, will aid in verifying the accuracy of the drawing.

When more than one tangent can be drawn parallel to HF, there will be more than one solution to the problem.

96. PROBLEM 25. **To pass a plane tangent to a right cylinder with a circular base having its axis parallel to the ground line, at a given point on the surface.**

Let GK, Fig. 50, be the axis of the cylinder, and p' the vertical projection of the point. To determine its horizontal projection, at p' erect a perpendicular to V. Through this pass the profile plane $d'Tc'$ (Art. 19). It intersects the cylinder in the circumference of a circle, of which M is the center. This circumference will intersect the perpendicular in two points of the surface. Revolve this plane about Tc' until it coincides with V (Arts. 19–22). M falls at m''. With m'' as a center, draw the revolved position of the circle cut from the cylinder; p_1'' and p_2'' will be the revolved positions of the two points, and p_1 and p_2 their horizontal projections. Let the plane be passed tangent to the cylinder at P_2.

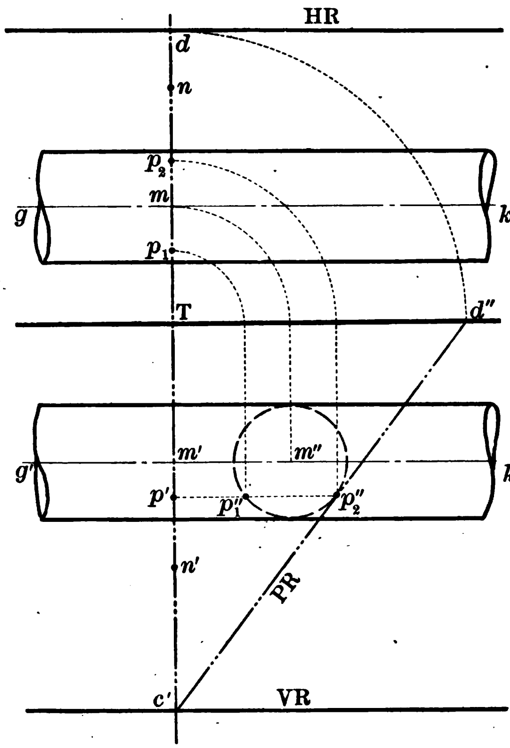


FIG. 50.

Analysis. Since the plane must contain a rectilinear element, it will be parallel to the ground line, and its traces,

therefore, parallel to the ground line (Prop. VIII, Art. 14). If through the given point a plane be passed perpendicular to the axis, and a tangent be drawn to its intersection with the cylinder at the point, it will be a line of the required plane. If through the points in which this line pierces the planes of projection lines be drawn parallel to the ground line, they will be the required traces.

Construction. Through P pass the plane T, and revolve it as above. At p_2'' draw $p_2''c'$ tangent to the circle; it is the revolved position of the tangent. When the plane is revolved to its primitive position, this tangent pierces V at c' and H at d, Td being equal to Td''; and HR and VR are the required traces.

97. PROBLEM 26. **To pass a plane tangent to a right cylinder with a circular base having its axis parallel to the ground line, through a given point without the surface.**

Let the cylinder be given as in Fig. 50, and let **N** be the given point.

Analysis. Since the required plane must contain a rectilinear element, it must be parallel to the ground line. If through the given point a plane be passed perpendicular to the axis, it will cut from the cylinder a circumference equal to the base; and if through the point a tangent be drawn to this circumference, it will be a line of the required plane, and the traces may be determined as in the preceding problem.

Since two tangents can be drawn, there may be two tangent planes.

Let the construction be made in accordance with the analysis.

98. PROBLEM 27. **To pass a plane parallel to a given straight line, and tangent to a right cylinder with a circular base having its axis parallel to the ground line.**

Let the cylinder be given as in Fig. 51, and let **MN** be the given line.

Analysis. Since the plane must be parallel to the axis, if

through the given line we pass a plane parallel to the axis, it will be parallel to the required plane and to the ground line.

A plane perpendicular to the ground line will cut from the cylinder a circumference; from the required tangent plane a

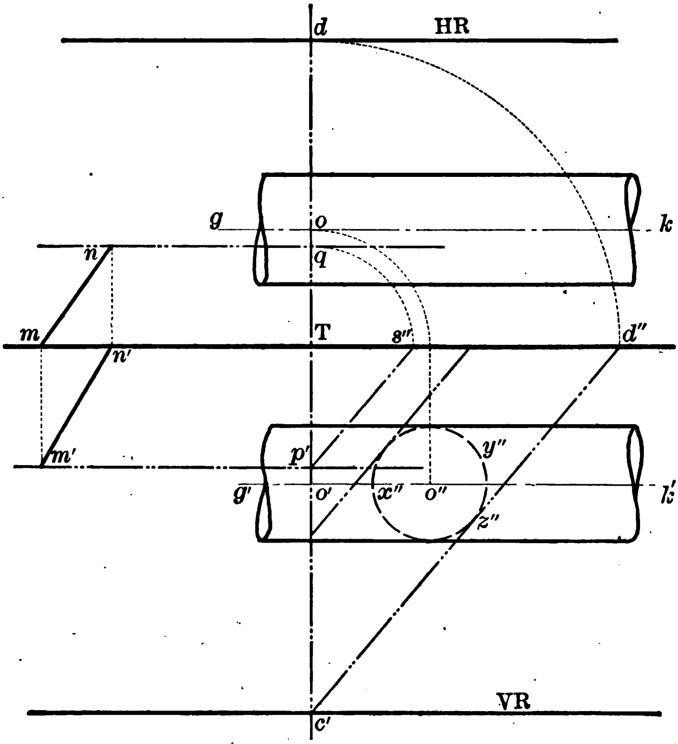


FIG. 51.

tangent to this circumference (Art. 86), and from the parallel plane a straight line parallel to the tangent. If, then, we construct the circumference, and draw to it a tangent parallel to the intersection of the parallel plane, this tangent will be a line of the required plane, from which the traces may be found as in the preceding problems.

Construction. Through m' and n draw the lines $m'p'$ and nq, parallel to the ground line. They will be the traces of the parallel plane (Art. 33). Let dTc' be the plane perpendicular to the ground line. It cuts from the cylinder a circle whose center is O, and from the parallel plane a straight line which pierces H at q and V at p' (Art. 40). Revolve this plane about Tc' until it coincides with V; $x''y''z''$ will be the revolved position of the circle, and $s''p'$ that of the line cut from the parallel plane; $c'd''$ tangent to $x''y''z''$, and parallel to $s''p'$, will be the revolved position of a line of the required plane, which, in its true position, pierces H at d, and V at c', and HR and VR are the traces of the required plane.

Since another parallel tangent can be drawn, there will be two solutions.

99. PROBLEM 28. **To pass a plane tangent to a cone, through a given point on the surface.**

Let the cone be given as in Fig. 48, and let P, assumed as in Art. 85, be the given point.

Analysis. The required plane must contain the rectilinear element, passing through the given point (Art. 88). If, then, we draw this element (Art. 85), and at the point where it pierces the horizontal plane draw a tangent to the base, it will be the horizontal trace of the required plane, and points of the vertical trace may be found as in the similar case for the cylinder.

Let the construction be made in accordance with the analysis.

If the base of the cone lies in an oblique plane, as in Fig. 48 *a*, the tangent QV will, of course, not be the horizontal trace. In this case the tangent plane is determined by QV and the element QS.

100. PROBLEM 29. **Through a point without the surface of a cone to pass a plane tangent to the cone.**

Let the cone be given as in Fig. 52, and let P be the given point.

Analysis. Since the required plane must contain a rectilinear element, it will pass through the vertex; hence, if we join the given point with the vertex by a straight line, it will be a line of the required plane, and pierce the horizontal plane in a point of the required horizontal trace, which may then be drawn tangent to the base. If the point of contact with the base be joined to the vertex by a straight line, it will be the element of contact. The vertical trace may be found as in the preceding problems.

If more than one tangent can be drawn to the base, there will be more than one solution.

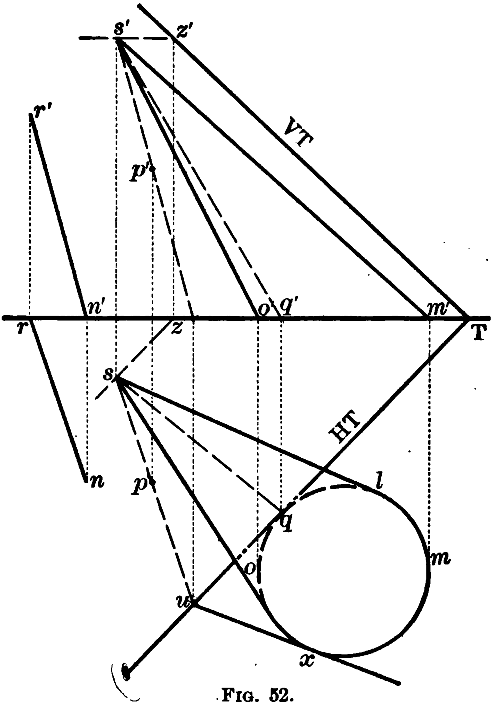


FIG. 52.

Construction. Join P with S by the line PS; it pierces H at u. Draw uq tangent to mlo; it is the required horizontal trace. SQ is the element of contact which pierces V in a point of the vertical trace. SZ, parallel to HT, pierces V at z', and z'T is the vertical trace.

ux will be the horizontal trace of a second tangent plane through P.

101. PROBLEM 30. **To pass a plane tangent to a cone and parallel to a given straight line.**

Let the cone be given as in Fig. 52, and let NR be the given straight line.

Analysis. If through the vertex we draw a line parallel to the given line, it must lie in the required plane, and pierce the horizontal plane in a point of the horizontal trace. Through this point draw a tangent to the base; it will be the required horizontal trace, and the element of contact and the vertical trace may be found as in the preceding problem.

When more than one tangent can be drawn to the base, there will be more than one solution.

If the parallel line through the vertex pierces the horizontal plane within the base, no tangent can be drawn, and the problem is impossible.

Let the construction be made in accordance with the analysis.

Ex. 97. Given a right circular cone with its axis parallel to the ground line; pass a plane tangent to the cone at a point on the surface.

Ex. 98. Pass a plane tangent to the above cone through a point outside the surface.

Ex. 99. Pass a plane tangent to the cone of Ex. 97 and parallel to a given straight line.

Ex. 100. Given a plane T, and a point P lying in that plane. Construct the projection of a line PX, passing through P, making an angle of 30° with H and lying in the plane T.

Ex. 101. Given an oblique plane and an oblique line, to pass another plane through the line making an angle of 45° with the given plane. Analyze and construct.

Points in which Surfaces are Pierced by Lines

102. The points in which a straight line pierces a surface are easily found by passing through the line any plane intersecting the surface. It will cut from the surface a line which will inter-

sect the given line in the required points. This auxiliary plane should be so chosen as to intersect the surface in the simplest line possible.

If the given surface be a cylinder, a plane through the straight line parallel to the rectilinear elements should be used. It will intersect the cylinder in one or more rectilinear elements, which will intersect the given line in the required points.

If the given surface be a cone, the auxiliary plane should pass through the vertex.

If the given surface be a sphere, the auxiliary plane should pass through the center, and the line of intersection will be a great circle of the sphere.

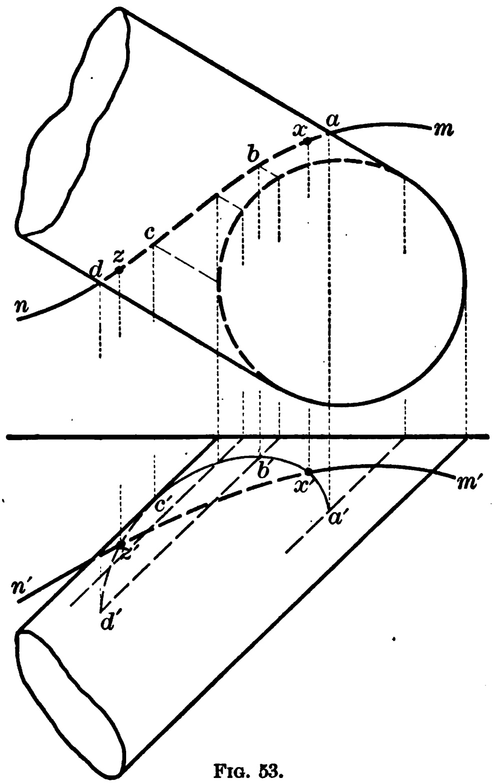


FIG. 53.

103. The points in which a *curved* line pierces a surface may be found by using the horizontal or vertical projecting cylinder (Art. 80) of the line. This will cut from the given surface a curved line which will contain the required piercing point. Thus, let it be required to find the points in which the line MN, Fig. 53, pierces the surface of the cylinder. The horizontal projecting

cylinder of MN cuts the given cylinder in a curve horizontally projected in *abc*, and vertically projected in $a'b'c'$. The line MN cuts this curve in the points X and Z, which are the required piercing points.

Ex. 102. Find the point in which a curved line pierces an oblique plane.

Ex. 103. Find the points in which a curved line pierces a cone.

INTERSECTION OF CYLINDERS AND CONES. DEVELOPMENTS

104. The solution of the problem of the intersection of surfaces consists in finding two lines, one on each surface, which intersect. The point of intersection will lie in both surfaces, and will therefore be a point of their line of intersection. A series of such points will determine the curve.

105. To find the intersection of a plane with a surface, we intersect the plane and surface by a system of auxiliary planes. Each plane will cut from the given plane a straight line, and from the surface a line, the *intersection of which will be points of the required line.* The system of auxiliary planes should be so chosen as to cut from the surface the simplest lines; a rectilinear element if possible, or the circumference of a circle, etc.

To draw a tangent to the curve of intersection at a given point. This tangent must lie in the intersecting plane, that is, in the plane of the curve (Art. 60). It must also lie in the plane tangent to the surface at the given point (Art. 86). Hence, *if we pass a plane tangent to the given surface at the given point and determine its intersection with the intersecting plane, this will be the required tangent.*

106. PROBLEM 31. **To find the intersection of a right cylinder with a circular base by a plane.**

Let *mlo*, Fig. 54, be the base of the cylinder, and *c* the horizontal, and $c'd'$ the vertical projection of the axis; then *mlo*

will be the horizontal, and $s'a'n'u'$ the vertical projection of the cylinder (Art. 81). Let T be the intersecting plane.

Analysis. Intersect the cylinder and plane by a system of auxiliary planes parallel to the axis and also to the horizontal trace of the given plane. These will cut from the cylinder rectilinear elements, and from the plane straight lines parallel to the horizontal trace. The intersection of these lines will be points of the required curve (Art. 105).

This curve, as also the intersection of any cylinder or cone with a circular or an elliptical base, by a plane cutting all the rectilinear elements, is an ellipse.

Construction. Draw xy parallel to HT; it will be the horizontal trace of one of the auxiliary planes; yy' is its vertical trace. It intersects the cylinder in two elements, one of which pierces H at x and the other at z, and $f'x'$ and $b'z'$ are their vertical projections. It intersects the plane in the straight line XY (Art. 40), and X and Z, the intersections of this line with the two elements, are points of the curve. In the same way any number of points may be found.

If a point of the curve on any particular element be required, we have simply to pass an auxiliary plane through this element. Thus, to construct the point on (o, $n'u'$), draw the trace ow, and construct as above the point O.

Since this point lies on the curve, and also on the extreme element, and since no point of the curve can be vertically projected outside of $n'u'$, the vertical projection of the curve must be tangent to $n'u'$ at o'. Or, the reason may be given thus, in many like cases: If a tangent be drawn to the curve at O, it will lie in the plane tangent to the surface at O (Art. 86); and since this tangent plane is perpendicular to the vertical plane, the tangent will be vertically projected into its trace $n'u'$, which must therefore be tangent to $m'o'l'$ at o' (Art. 61). Also $a's'$ must be tangent to $m'o'l'$ at m'.

If a plane be passed through the axis perpendicular to the

intersecting plane, it will evidently cut from the plane a straight line which will bisect all the chords of the curve perpendicular to it, and this line will be the transverse axis of the ellipse; *lc* is the horizontal trace of such a plane, and *gg'* the vertical. It cuts the given plane in KG (Art. 40), and the cylinder in two elements horizontally projected at *k* and *l*, and KL is the transverse axis, and K and L are the vertices of the ellipse. K is the lowest, and L the highest, point of the curve.

Since the curve lies on the surface of the cylinder, its horizontal projection will be in the base *mlo*.

To draw a tangent to the curve at any point, as X; draw *xr* tangent to *xol* at *x*; it is the horizontal trace of a plane tangent to the cylinder at X (Art. 93). This plane intersects the plane T in a straight line, which pierces H at *r*; and since X is also a point of the intersection, RX will be the required tangent (Art. 105). If a tangent be drawn at each of the points determined as above, the projections of the curve can be drawn, with great accuracy, tangent to the projections of these tangents, at the projections of the points of tangency.

The part of the curve MXO, between the points M and O, lies in front of the extreme elements (m, $a's'$) and (o, $n'u'$) and is seen, and therefore $m'x'o'$ is drawn full, and $m'l'o'$ broken.

To represent the curve in its true dimensions, let its plane be revolved about HT until it coincides with H. The revolved position of each point, as X, at x_1, will be determined as in Art. 34; k_1 and l_1 will be the revolved positions of the vertices, rx_1 will be the revolved position of the tangent, and $l_1x_1k_1$ the ellipse in its true size.

107. Development. Since the plane tangent to a single-curved surface in general contains two consecutive rectilinear elements (Art. 90), it will contain the elementary portion of the surface generated by the generatrix in moving from the first element to the second. Now if the surface be rolled over

until the consecutive element following the second comes into the tangent plane, the portion of the plane limited by the first and third elements will equal the portion of the surface limited

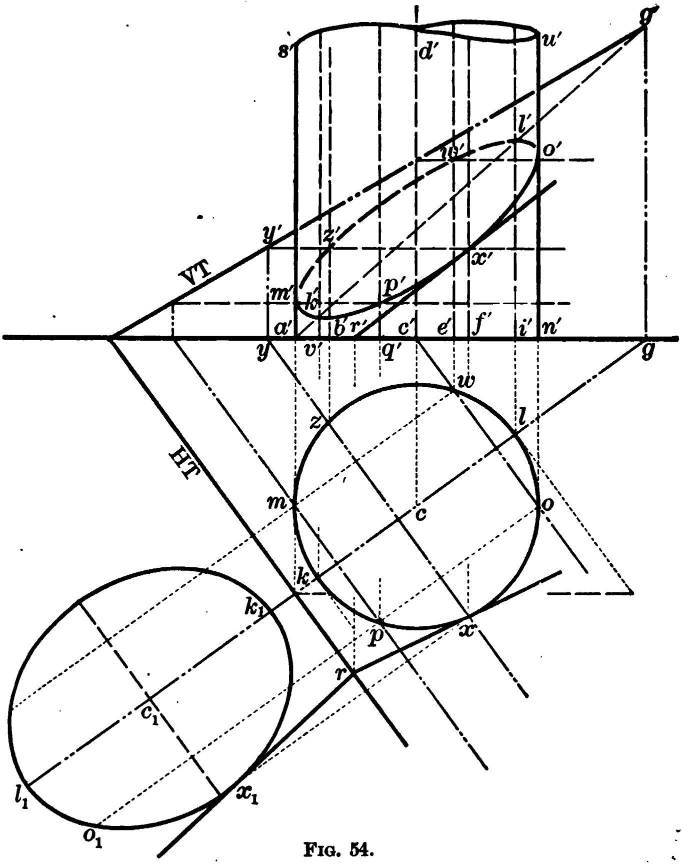


FIG. 54.

by the same elements. If we continue to roll the surface on the tangent plane until any following element comes into it, the portion of the plane included between this element and the first will be equal to the portion of the surface limited by the same elements. Therefore, if a single-curved surface be rolled

over on any one of its tangent planes until each of its rectilinear elements has come into this plane, ***the portion of the plane thus touched by the surface, and limited by the extreme elements,*** will be a plane surface equal to the given surface, and is ***the development of the surface.***

As a plane tangent to a warped surface cannot contain two consecutive rectilinear elements (Art. 140), the elementary surface limited by these two elements cannot be brought into a plane without breaking the continuity of the surface. **A warped surface, therefore, cannot be developed.**

Neither can a double-curved surface be developed, as any elementary portion of the surface will be limited by two curves, and cannot be brought into a plane without breaking the continuity of the surface.

108. In order to determine the position of the different rectilinear elements of a single-curved surface, as they come into the tangent plane, or *plane of development*, it will always be necessary to find some curve upon the surface which will develop into a straight line, or circle, or some simple known curve, upon which the rectified distances between these elements can be laid off.

109. PROBLEM 32. **To develop a right cylinder with a circular base, and trace upon the development the curve of intersection of the cylinder by an oblique plane.**

Let the cylinder and curve be given as in the preceding problem, and let the plane of development be the tangent plane at K.

Analysis. Since the plane of the base is perpendicular to the element of contact of the tangent plane, it is evident that as the cylinder is rolled out on this plane, this base will develop into a straight line on which we can lay off the rectified distances between the several elements and then draw them each parallel to the element of contact.

Points of the development of the curve are found by laying off on the development of each element, from the point where it meets the rectified base, a distance equal to the distance of the point from the base.

Construction. The plane of development being coincident with the plane of the paper, let kK, Fig. 54 *a*, be the element of

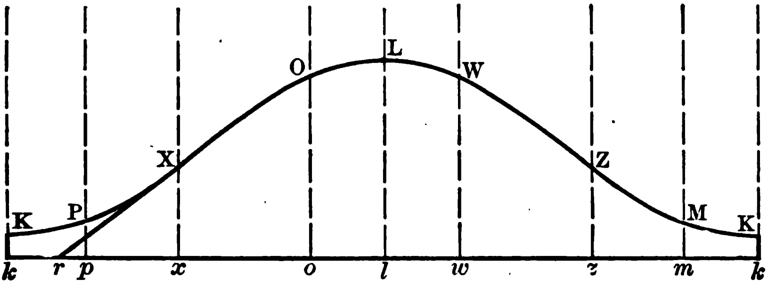


Fig. 54 *a*.

contact. Draw kk perpendicular to kK, and lay off kk equal to the rectified circumference $kxwk$, Fig. 54. It will be the development of the base. Lay off kp equal to the arc kp, and draw pP parallel to kK. It is the development of the element which pierces H at p. Likewise for each of the elements lay off px, xo, etc., equal respectively to the rectified arcs px, xo, etc., in Fig. 54, and draw xX, oO, etc. The portion of the plane included between kK and kK will be the development of the cylinder.

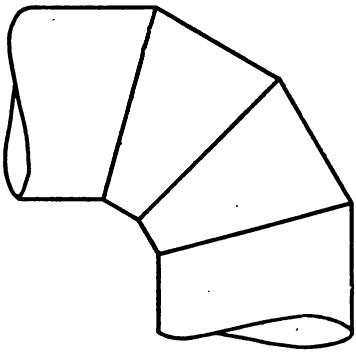


Fig. 55.—Stovepipe Elbow.

On kK lay off kK equal to $v'k'$, and K will be one point of the developed curve; on pP lay off pP equal to $q'p'$, and P will be a second point; and thus each point may be determined, and K - - L - - K will be

the developed curve. The line rx, the *sub-tangent*, will take the position rx on the developed base, and Xr will be the tangent at X in the plane of development. This must be tangent at X, since the tangent after development must contain the same two consecutive points which it contains in space, and therefore be tangent to the development of the curve (Art. 60).

110. Problem 33. **To find the intersection of an oblique cylinder by a plane.**

Let the cylinder be given as in Fig. 56, and let T be the intersecting plane.

Analysis. Intersect the cylinder and plane by a system of auxiliary planes parallel to the rectilinear elements and perpendicular to the horizontal plane. These planes will each intersect the cylinder in two rectilinear elements, and the plane in a straight line, the intersection of which will be points of the curve.

Construction. Let eq, parallel to li, be the horizontal trace of an auxiliary plane; qq' will be its vertical trace. It intersects the cylinder in two elements, one of which pierces H at r, the other at s, and these are vertically projected in $r'g'$ and $s'h'$, and horizontally in eq. It intersects the plane T in a straight line, which pierces H at e, and is vertically projected in $e'q'$, and horizontally in eq. These lines intersect in Z and Y (Prop. XXVIII, Art. 16), points of the curve. In the same way any number of points may be determined.

The auxiliary planes, being parallel, must intersect T in parallel lines, the vertical projections of which will be parallel to $e'q'$.

By the plane whose horizontal trace is mn, the points U and X are determined. The vertical projection of the curve must be tangent to $m'n'$ at u'. The points in which the horizontal projection is tangent to li and kf are determined by using these lines as the traces of auxiliary planes.

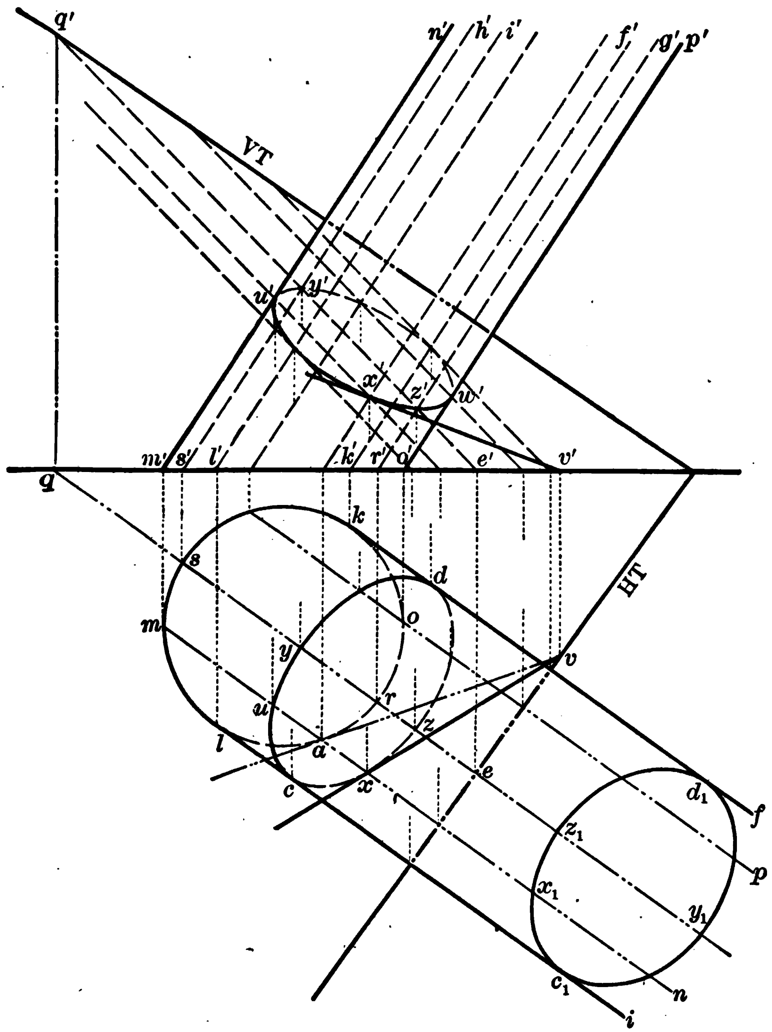


FIG. 56.

To draw a tangent to the curve at any point, as X, pass a plane tangent to the cylinder at X; *av* is its horizontal trace, and *vx* the horizontal, and *v'x'* the vertical projection of the tangent.

A sufficient number of points and tangents being thus determined, the projections of the curve can be drawn with accuracy. The part *cyd* is full, being the horizontal projection of that part of the curve which lies above the extreme elements LI and KF. For a similar reason, $u'x'w'$ is full.

To show the curve in its true dimensions, revolve the plane about HT until it coincides with H. The revolved position of each point may be found as in Art. 34, and $c_1y_1d_1z_1$ will be the curve in its true size.

The section in this case is a ***right section*,** **since the cutting plane was assumed perpendicular to the elements.**

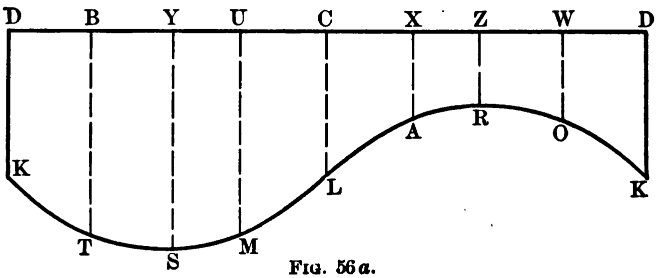


FIG. 56 *a*.

111. If it be required to develop the cylinder on a tangent plane along any element, as KDF, we first make a right section as above. We know this will develop into a straight line perpendicular to KDF. On this we lay off the rectified arcs of the section included between the several elements, and then draw these elements parallel to KD.

The developed base, *or any curve on the surface*, may be traced on the plane of development, Fig. 56 *a*, by laying off on each element, from the developed position of the point where it intersects the right section, the distance from this point to the point where the element intersects the base or curve. A line through the extremities of these distances will be the required development.

Ex. 104. Pass a plane perpendicular to the elements of an oblique cylinder; find the curve of right section, and then develop that portion of the cylindrical surface lying between the plane of right section and the base.

112. PROBLEM 34. **To find the intersection of a right cone with a circular base, by a plane.**

Let the cone be given as in Fig. 57, and let T be the given plane, the vertical plane being assumed perpendicular to it.

Analysis. Intersect the cone by a system of planes through the vertex and perpendicular to the vertical plane. The element cut from the cone by each plane (Art. 105) will intersect the straight line cut from the given plane in points of the required curve.

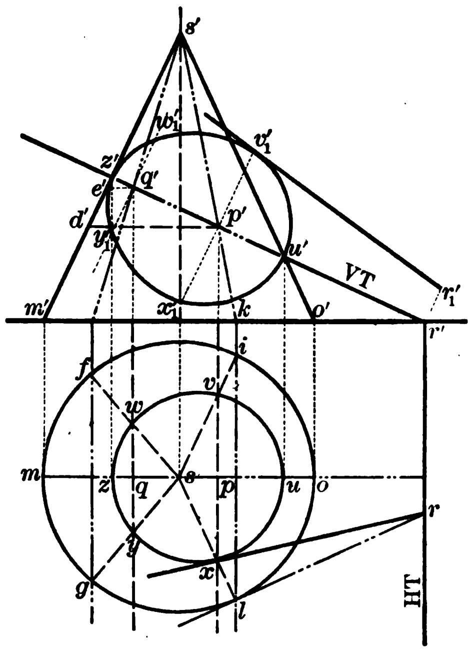


FIG. 57.

Construction. Let lk be the horizontal trace of an auxiliary plane; ks' will be its vertical trace. It intersects the cone in two elements, one of which pierces H in l, and the other in i, horizontally projected in ls and is respectively. It intersects the given plane in a straight line perpendicular to V, vertically projected at p', and horizontally in xv; hence x and v are the horizontal projections of two points of the curve, both vertically projected at p'.

In the same way any number of points can be found, as yq', wq', etc.

The plane whose horizontal trace is *mo*, perpendicular to T, intersects it in a straight line vertically projected in VT, which evidently bisects all chords of the curve perpendicular to it, and is therefore an axis of the curve. This plane cuts from the cone the elements SM and SO, which are intersected by the axis in the points Z and U, which are the vertices.

To draw a tangent to the curve at any point, as X, pass a plane tangent to the cone at X; *lr* is its horizontal trace. It intersects T in (rx, $r'p'$), which is therefore the required tangent (Art. 105). The horizontal projection, *uxzv*, can now be drawn. $u'z'$ is its vertical projection.

To represent the curve in its true dimensions, we may revolve it about HT until it coincides with H, or about VT until it coincides with V, and determine it as in Art. 34. Otherwise, thus: Revolve it about UZ until its plane becomes parallel to V; it will then be vertically projected in its true dimensions (Art. 58). The points U and Z being in the axis, will be projected at u' and z' respectively. X will be vertically projected at x'_1, $p'x'_1$ being equal to px; Y at y'_1, $q'y'_1$ being equal to qy; W at w'_1, etc.; and $u'v'_1z'x'_1$ will be the curve in its true size.

113. If a right cone with a circular base be intersected by a plane, as in Fig. 57, making a *less angle* with the plane of the base than the elements do, the curve of intersection is an *ellipse*. If it make the *same angle*, or is parallel to one of the elements, the curve is a *parabola*. If it make a *greater angle*, the curve is a *hyperbola*. Hence these three curves are known by the general name *conic sections*.

114. PROBLEM 35. **To develop a right cone with a circular base.**

Let the cone and its intersection by an oblique plane be

given as in the preceding problem, Fig. 57, and let the plane of development be the plane tangent along the element MS; the half of the cone in front being rolled to the left, and the other half to the right.

Analysis. Since the base of the cone is everywhere equally distant from the vertex, as the cone is rolled out, each point of this base will be in the circumference of a circle described with the vertex as a center, and a radius equal to the distance

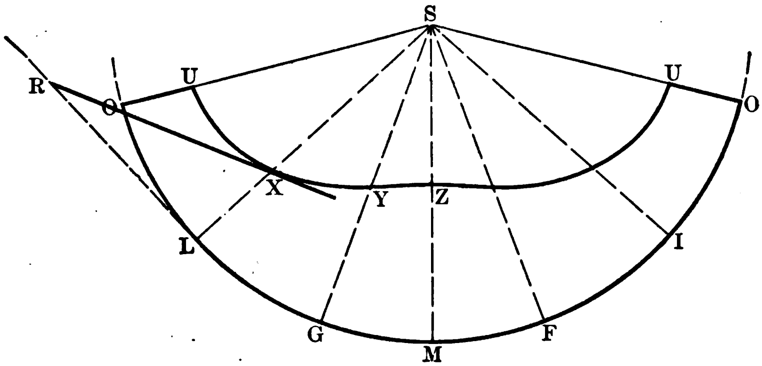


FIG. 57 *a*.

from the vertex to any point of the base. By laying off on this circumference the rectified arc of the base, contained between any two elements, and drawing straight lines from the extremities to the vertex, we have, in the plane of development, the position of these elements. Laying off on the proper elements the distances from the vertex to the different points of the curve of intersection, and tracing a curve through the extremities, we have the development of the curve of intersection.

Construction. With SM equal to $s'm'$, Figs. 57 and 57 *a*, describe the arc OMO. It is the development of the base. Lay off MG equal to *mg*, and draw SG; it is the developed position of the element SG. In the same way lay off GL equal to *gl*, LO equal to *lo*, and draw SL and SO. OSM is the develop-

ment of one half the cone. In like manner the other half may be developed.

On SM lay off SZ equal to $s'z'$. Z will be the position of the point in the plane of development. To obtain the true distance from S to Y, revolve SY about the axis of the cone until it becomes parallel to V, as in Art. 26; $s'e'$ will be its true length. On SG lay off SY equal to $s'e'$; on SL, SX equal to $s'd'$; on SO, SU equal to $s'u'$. Z, Y, X, and U will be the positions of these points on the plane of development, and UX - - U will be the development of the curve of intersection.

Through L draw LR perpendicular to LS, and make it equal to lr. RX will be the developed tangent.

Note. The work of rectifying the circumference of the base can be more accurately done by calculating the angle OSO from the relation,

$$\text{Angle OSO} = \frac{360\ \text{R}}{\sqrt{h^2 + \text{R}^2}}\ \textit{degrees},$$

where R is the radius of the base of the cone, and h is the altitude. Furthermore, if in Fig. 57 the base had been divided into equal parts before passing the auxiliary planes, the arcs mg, gl, etc., would all be equal, and their lengths would be laid off in Fig. 57 a by simply dividing OMO into the same number of equal parts.

115. PROBLEM 36. **To find the intersection of any cone by a plane.**

Let the cone and plane T be given as in Fig. 58.

Analysis. Intersect the cone by a system of planes through the vertex and perpendicular to the plane of the base. Each of these planes will intersect the cone in one or more rectilinear elements, and the given plane in a straight line, the intersection of which will be points of the curve. Since these auxiliary planes are perpendicular to the plane of the base

they will intersect in a straight line through the vertex perpendicular to the plane of the base, and the point in which this line pierces the cutting plane will be a point common to all the straight lines cut from this plane.

Construction. Find the point in which the perpendicular through S to the plane of the base (H in this problem) pierces T, as in Art. 28. v' is its vertical projection. The vertical projections of the lines cut from T all pass through this point.

Let sp be the horizontal trace of an auxiliary plane. It intersects the cone in the elements SE and SD, and the cutting plane in the straight line (ps, $p'v'$). This line intersects the elements in R and Y, which are points of the required curve. In the same way any number of points may be found.

To find the point of the curve on any particular element, as SM, we pass an auxiliary plane through this element. sm is its horizontal trace, and Z is the point on this element. The vertical projection of the curve is tangent to $s'm'$ at z', and to $s'o'$ at u'. The points q and w, in which the horizontal projection is tangent to sl and sn, are found by using as auxiliary planes the two planes whose traces are sl and sn.

To draw a tangent to the curve at any point, as X, pass a plane tangent to the cone at X. ic is its horizontal trace (Art. 99). It intersects T in CX, which is therefore the required tangent (Art. 105).

The part of the curve which lies above the two extreme elements SL and SN is seen, and therefore its projection, $wyzq$, is full. For a similar reason the projection $z'q'x'u'$, of that part of the curve which lies in front of the two extreme elements, SM and SO, is full.

To show the curve in its true dimensions, revolve the plane about HT until it coincides with H, and determine each point, as Q at q_1, as in Art. 34. Or the position of (s, v') may be found

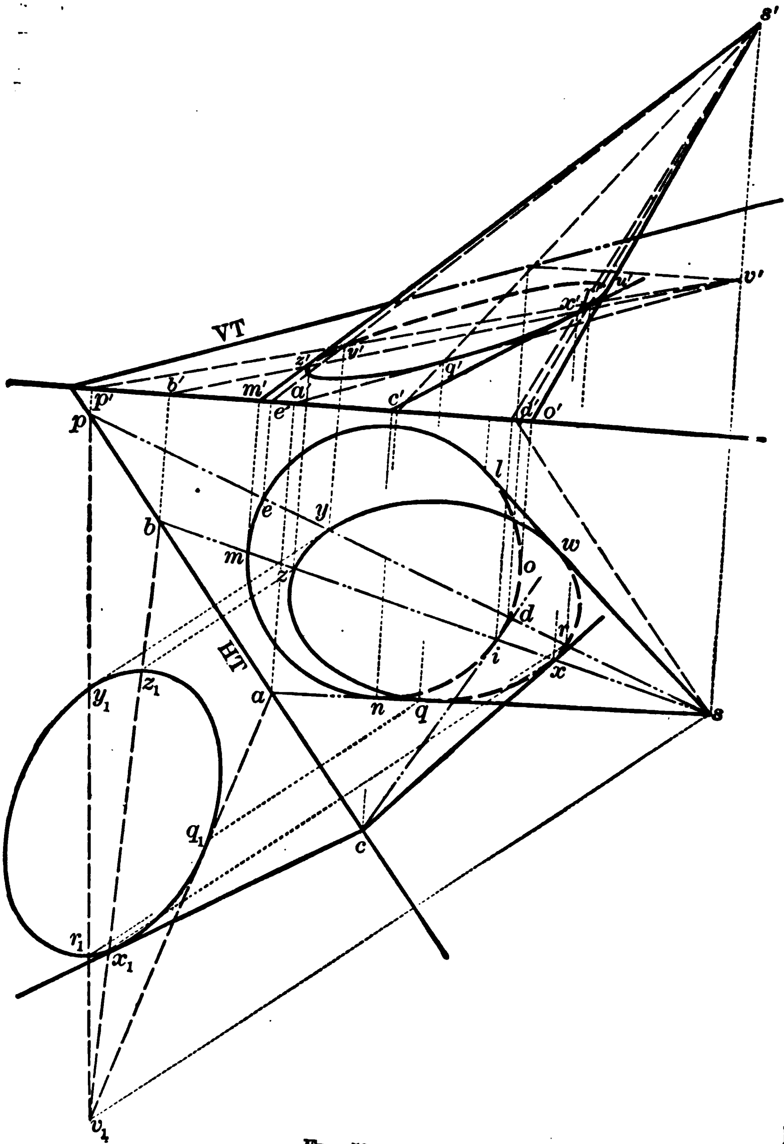


Fig. 58.

at v_1. Then if the points c, a, b, p, etc., be each joined with this point by straight lines, we have the revolved positions of the lines cut from the given plane by the auxiliary planes, and the points y_1, z_1, r_1, x_1, in which these lines are intersected by the perpendiculars to the axis, yy_1, zz_1, etc., are points of revolved position of the curve. x_1c is the revolved position of the tangent.

If, in Fig. 57, both traces of the cutting plane made acute angles with the ground line, how would the auxiliary planes be passed in order to find the required points most conveniently?

116. To find the intersection of any two curved surfaces, we intersect them by a system of auxiliary surfaces. Each auxiliary surface will cut from the given surfaces lines, *the intersection of which will be points of the required line* (Art. 104).

The system of auxiliary surfaces should be so chosen as to cut from the given surfaces the simplest lines, rectilinear elements if possible, or the circumferences of circles, etc.

To draw a tangent to the curve of intersection at any point, pass a plane tangent to each surface at this point. The *intersection* of these two planes will be the required *tangent,* since it must lie in each of the tangent planes (Art. 86).

In constructing this curve of intersection, great care should be taken to determine those points in which its projections are tangent to the limiting lines of the projections of the surfaces; and also those points in which the curve itself is tangent to other lines of either surface, as these points aid much in drawing the curve with accuracy.

117. PROBLEM 37. **To find the intersection of a cylinder and a cone.**

Let the surfaces be given as in Fig. 59; the base of the cylinder bci being in the horizontal plane, the base of the cone $m'l'o'$ in the vertical plane, and S its vertex.

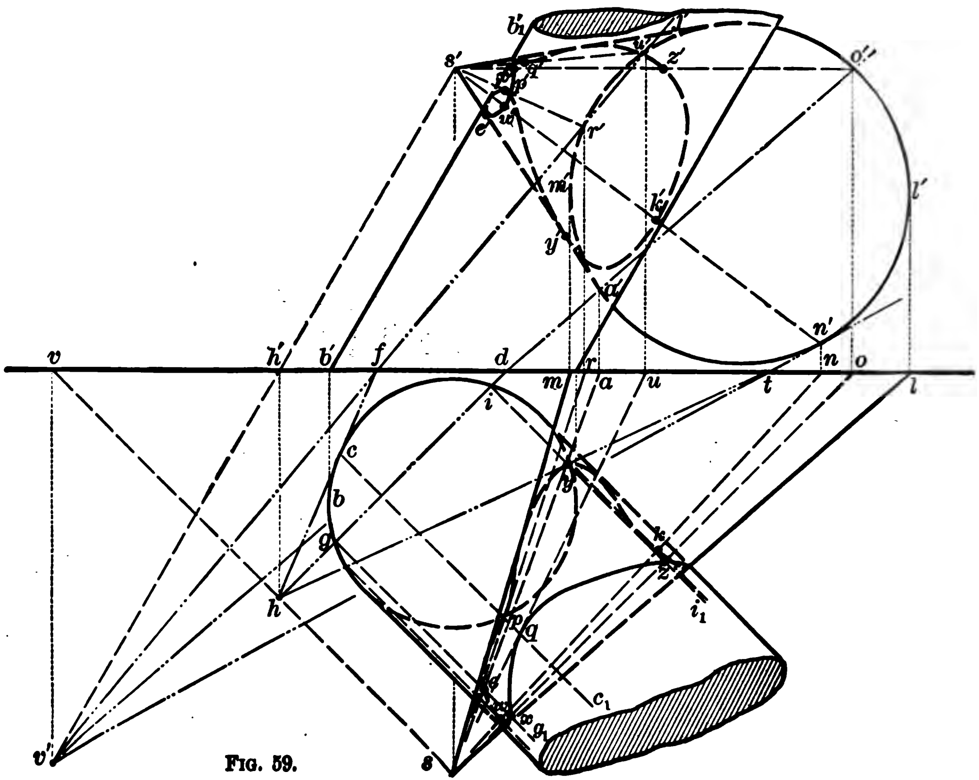


FIG. 59.

Analysis. Intersect the two surfaces by a system of auxiliary planes ***passing through the vertex of the cone and parallel to the rectilinear elements of the cylinder.*** Each plane will intersect each of the surfaces in two rectilinear elements, the intersection of which will be points of the required curve. These planes will intersect in a straight line passing through the vertex of the cone and parallel to the rectilinear elements of the cylinder, and the point in which this line pierces the horizontal plane will be a point common to the horizontal traces of all the auxiliary planes. Also the point in which this line pierces V will be common to the vertical traces of all the auxiliary planes.

Construction. Through S draw SV parallel to the elements of the cylinder. It pierces H in h, and V in v'. Through h draw any straight line, as hd; it may be taken as the horizontal trace of an auxiliary plane, the vertical trace of which is do', passing through v', the point in which SV pierces the vertical plane. This plane intersects the cylinder in two elements which pierce H at g and i, and are horizontally projected in gg_1 and ii_1. The same plane intersects the cone in two elements which pierce V in a' and o', and are horizontally projected in as and os. These elements intersect in the points E, X, Y, and Z, which are points of the required curve. In the same way any number of points may be determined.

The vertical projection of the required curve is tangent to $a's'$ at the points e' and y'; since $a's'$ is the vertical projection of one of the extreme elements of the cone. The points of tangency on any other extreme element, either of the cone or the cylinder, may be determined by using an auxiliary plane which shall contain that element.

A tangent to the curve at any of the points thus determined may be constructed by finding the intersection of two planes, one tangent to the cylinder and the other to the cone, at this point (Art. 116).

The plane of which *hc* is the horizontal trace is tangent to the cylinder along the element CC_1, and intersects the cone in the two elements SR and SU. If at P a plane be passed tangent to the cone, it will be tangent along SP, which is also its intersection with the plane tangent to the cylinder. SP is then tangent to the curve at P, and for a similar reason SQ is tangent at Q. Hence the two projections *sp* and *sq* are tangent to *epy - - zq* at *p* and *q* respectively; and the vertical projections of the same elements will be tangent at p' and q'.

The plane of which $v't$ is the vertical trace is tangent to the cone along the element SN, and intersects the cylinder in two elements, the horizontal projections of which are tangent to *epy - - zq* at *k* and *w*.

The projections of the curve can now be drawn with great accuracy. The horizontal projection of that part which lies on the upper portion of both cylinder and cone is drawn full. Likewise the vertical projection of that part which lies on the front side of both surfaces is full.

If two of the auxiliary planes used in finding the line of intersection be passed tangent to the cylinder, and both intersect the cone, it is evident that the cylinder will penetrate the cone so as to form two distinct curves of intersection. If one intersects the cone and the other does not, a portion only of the cylinder enters the cone, and there will be a continuous curve of intersection, as in the figure.

If neither of these planes intersects the cone, and the cone lies between them, the cone will penetrate the cylinder, making two distinct curves.

If both planes are tangent to the cone, all the rectilinear elements of both surfaces will be cut.

Ex. 105. Assume a cylinder and cone with bases in H, and find the curve of intersection, showing invisible portions in each projection by dashed lines.

118. PROBLEM 38. **To find the intersection of two cylinders.** If we pass a plane through a rectilinear element of one cylinder parallel to the rectilinear elements of the other, and then intersect the cylinders by a system of planes parallel to this plane, points on the curve of intersection of the two cylinders may be found by a process similar to that of the preceding problem. The horizontal traces of all the auxiliary planes will be parallel; likewise the vertical traces.

Ex. 106. Assume a cylinder with base in H, and another with base in V, and find the curve of intersection.

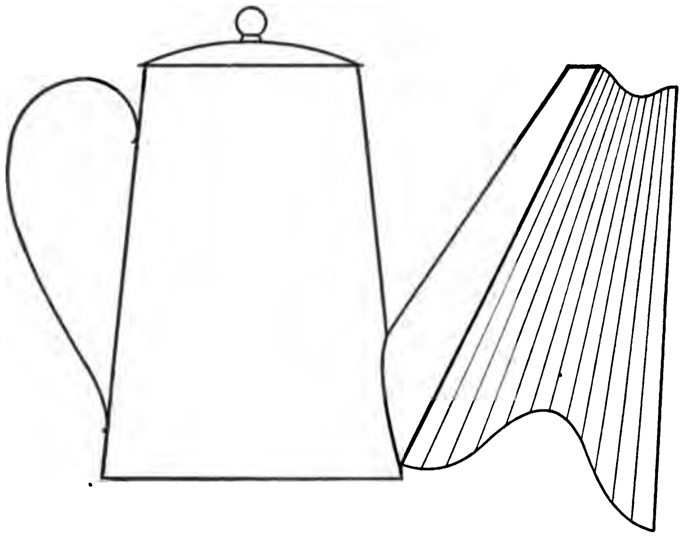


FIG. 60. — Intersection of Two Conical Surfaces; Development of Spout of Teapot.

119. PROBLEM 39. **To find the intersection of two cones,** pass a system of planes through the vertices of both cones. The straight line joining these vertices will lie in all of these planes, and pierce the horizontal plane in a point common to all the horizontal traces; and it will pierce the vertical plane in a point common to all the vertical traces.

We may ascertain whether the surfaces intersect in two distinct curves, or only one, in the same manner as in Art. **117**, by passing certain auxiliary planes tangent to either surface.

Ex. 107. Assume two intersecting cones, one with base in H, the other with base in V, and find the curve of intersection.

120. PROBLEM 40. **To find the intersection of a cone and a hemisphere.**

Let *mlo*, Fig. 61, be the base of the cone, and S its vertex at the center of the sphere, *abc* being the horizontal and $a'd'c'$ the vertical projection of the hemisphere.

Analysis. Intersect the surfaces by a system of planes passing through the vertex and perpendicular to the horizontal plane. Each plane will cut from the cone two rectilinear elements, and from the hemisphere a semicircumference, the intersection of which will be points of the required curve.

Construction. Take *sp* as the horizontal trace of one of the auxiliary planes. It intersects the cone in the two elements SP and SQ, and the hemisphere in a semicircle whose center is at S. Revolve this plane about the horizontal projecting line of S, until it becomes parallel to V. The element SP will be vertically projected in $s'p'_1$, SQ in $s'q'_1$, the semicircle in $a'd'c'$, and x'_1 and y'_1 will be the vertical projections of the revolved positions of the points of intersection. In the counter-revolution these points describe the arcs of horizontal circles and in their true position will be vertically projected at x' and y', and horizontally at x and y.

In the same way any number of points may be found.

The points of tangency u' and z' are found by using auxiliary planes which cut out the extreme elements SM and SO of the cone.

The points in which the vertical projection of the curve is tangent to the semicircle $a'd'c'$ are found by using the auxiliary plane whose trace is *st*.

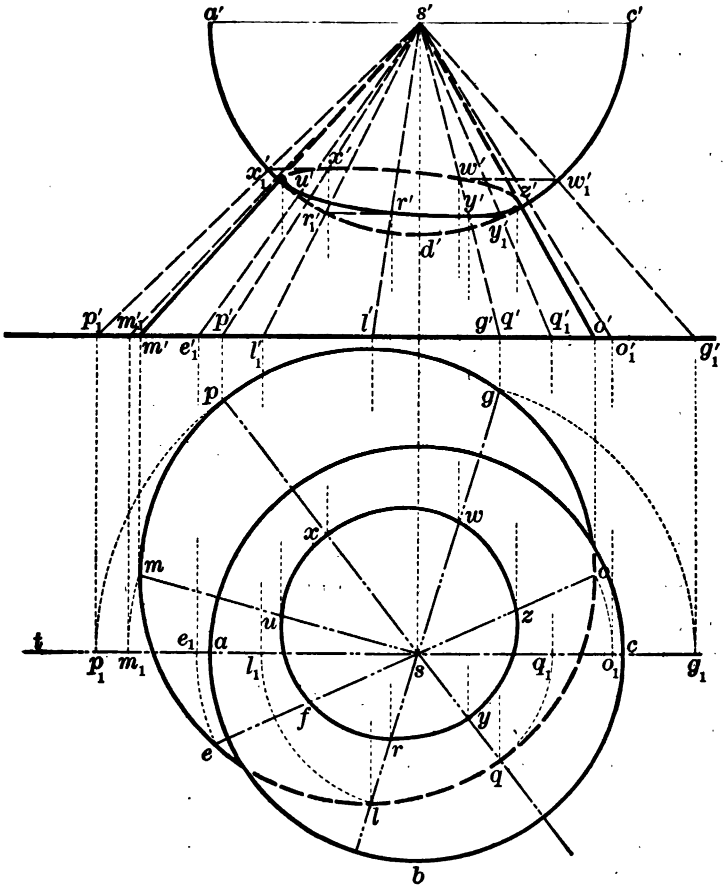


Fig. 61.

121. Problem 41. **To develop an oblique cone with any base.**

Let the cone be given as in the preceding problem, Fig. 61, and let it be developed on the plane tangent along the element SP.

First method. Analysis. If the cone be intersected by a sphere having its center at the vertex, all the points of the

curve of intersection will be at a distance from the vertex equal to the radius of the sphere; hence when the cone is developed this curve will develop into the arc of a circle having its center at the position of the vertex and its radius equal to that of the sphere. On this we can lay off the rectified arcs of the curve of intersection included between the several rectilinear elements (Art. 107), and then draw these elements to the position of the vertex.

The developed base, *or any curve on the surface*, may be traced on the plane of development by laying off on each element, from the vertex, the distance from the vertex to the point where the element intersects the base or curve. A line through the extremities of these distances will be the required development.

Construction. Find, as in the preceding problem, the curve XUY - - -. With S, Fig. 62, as a center and *sa* as a radius, describe the arc XUR - - - . It is the indefinite development of the intersection of the sphere and the cone. Draw SX for the position of the element SX.

To find the distance between any two points measured on the curve XUY - - -, we first develop its horizontal projecting cylinder on a plane tangent to it at X, as in Art. 109. X′U′R′X′, Fig. 62 *a*, is the development of the curve. On XUR lay off XU equal to X′U′, UR equal to U′R′, etc., and draw SU, SR, etc. These will be the positions of the elements on the plane of development. On these lay off SP equal to $s'p'_1$, SM equal to $s'm'_1$, etc., and join the points PME, etc., and we have the development of the base of the cone.

122. Second method; development by triangulation. If in Fig. 61 the elements are taken sufficiently numerous so that the chords *pm*, *me*, etc., are practically equal to the subtended arcs, the cone may be developed without the use of the auxiliary sphere, by conceiving the surface to be approximately a pyramid made up of the triangles PSM, MSE, ESL, etc.

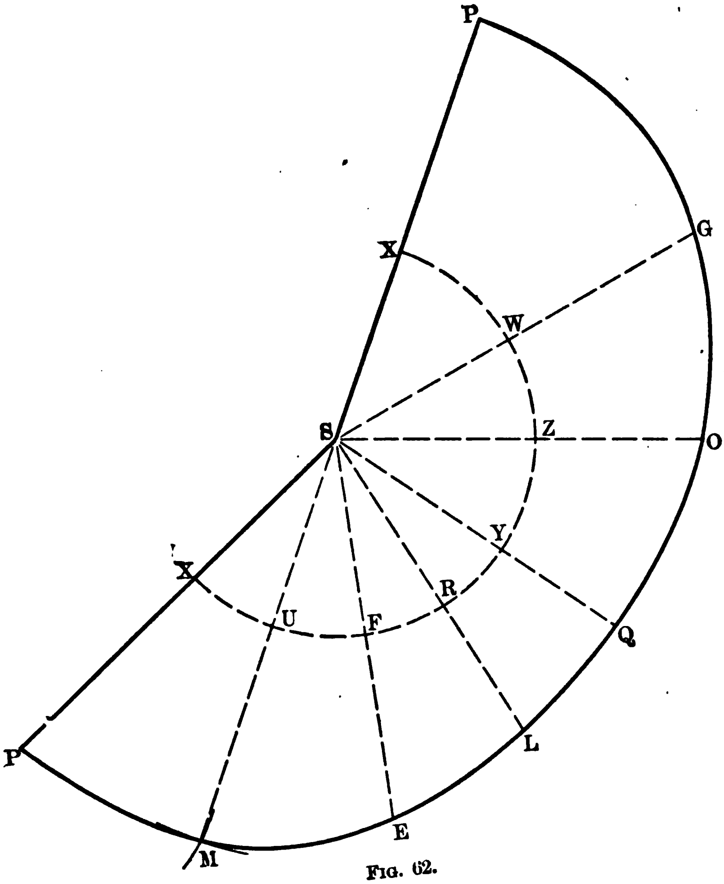


FIG. 62.

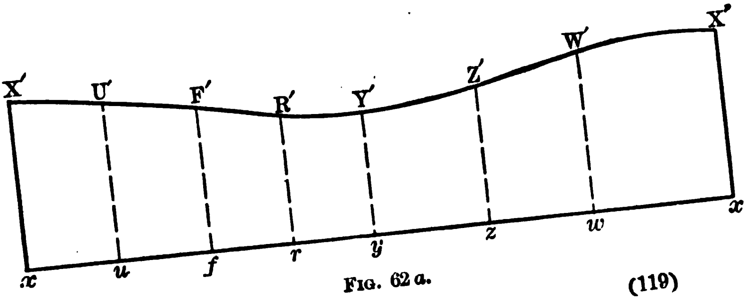


FIG. 62 *a*.

Let SP be laid off in its true length ($s'p'_1$). Then with S, Fig. 62, as center and a radius equal to $s'm'_1$, the true length of the element SM, strike an arc. Also with P as center and a radius equal to the chord pm, strike an arc intersecting the arc just drawn. This determines a triangle PSM. To the right of this triangle lay off the adjoining one SME in the same manner, continuing until all the triangles have been laid off. A smooth curve through the points PMEL, etc., will be the developed curve of the base.

Any curve of the surface may be traced on the plane of development by laying off on each element, from the vertex, the distance from the vertex to the point where the element intersects the curve, and drawing a smooth curve through the points thus determined.

Ex. 108. Analyze the problem, to find the shortest path on the surface of a cylinder or cone, between two given points on the surface.

Convolutes

123. Single-curved surfaces of the third kind are called convolutes. They may be generated by drawing a system of tangents to any space curve. These tangents will evidently be rectilinear elements of a single-curved surface. For if we conceive a series of consecutive points of a space curve as a, b, c, d, etc., the tangent which contains a and b (Art. 60) is intersected by the one which contains b and c at b; that which contains b and c, by the one which contains c and d at c; and so on, each tangent intersecting the preceding consecutive one, but not the others, since no two elements of the curve not consecutive are in general in the same plane (Art. 55).

124. The helical convolute. If the curve to which the tangents are drawn is a helix, the surface may be represented thus: Let pxy, Fig. 64, be the horizontal, and $p'x'y'$ the vertical pro-

jection of the helical directrix. Since the rectilinear elements are all tangent to this directrix, any one may be assumed, as in Art. 73; hence XZ, YU, etc., are elements of the surface.

A point of the surface may be assumed by taking a point on any assumed element.

These elements pierce H in the points *z*, *u*, *v*, etc., and *zuvw* is the horizontal trace of the surface, and may be regarded as

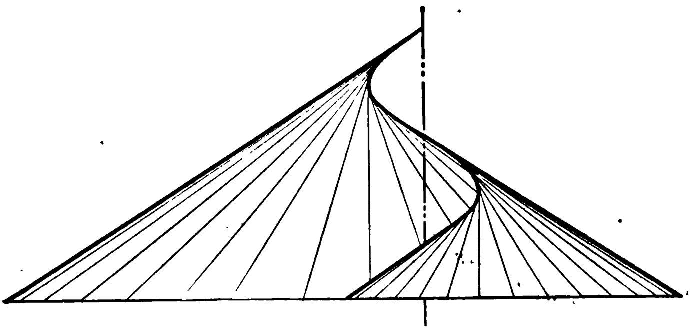


FIG. 63.—Helical Convolute.

its base; and it is evident that if the surface be intersected by any plane parallel to this base, the curve of intersection will be equal to the base.

Since each of the tangents, *xz*, *os*, etc., is equal to the rectified arc, from the springing point *p*, the curve is an *involute* of the circle *pxy*.

This surface is also known as the *Olivier Spiraloid*, and is used as the working surface in the teeth of skew bevel gears. See MacCord's Kinematics of Machines, Arts. 382–385; Robinson's Principles of Mechanism, pp 181–186; Grant's Teeth of Wheels, Arts. 175–177; American Machinist, Aug. 28, 1890.

125. PROBLEM 42. **To pass a plane tangent to a helical convolute at a given point on its surface.**

Let the surface be given as in Fig. 64, and let R be the given point (Art. 124).

Analysis. Since this is a single-curved surface, the tangent plane must be tangent all along the rectilinear element through the point of contact (Art. 89). Its horizontal trace must therefore be tangent to the base (Art. 91). If through the point where this element pierces the horizontal plane a straight line is drawn tangent to the base, it will be the required horizontal trace, and the vertical trace may be determined as in the case of the cylinder or cone.

Construction. The element RX pierces H at Z. At this point draw the tangent HT (perpendicular to *zx*); it is the horizontal trace. VT is the vertical trace.

126. PROBLEM 43. **To pass a plane tangent to a helical convolute through a given point without the surface,** we pass a plane through the point parallel to the base and draw a tangent to the curve of intersection (Art. 124) through the point. This tangent, with the element of the surface through its point of contact, will determine the tangent plane.

127. PROBLEM 44. **To pass a plane tangent to a helical convolute and parallel to a given straight line.**

Let the surface be given as in Fig. 64, and let MN be the given line.

Analysis. If with any point of the straight line as a vertex we construct a cone whose elements make the same angle with the horizontal plane as the elements of the surface, and pass a plane through the line tangent to this cone, it will be parallel to the required plane. The traces of the required plane may then be constructed as in Art. 95.

Construction. Take n' as the vertex of the auxiliary cone and draw $n'b$, making with the ground line an angle equal to $v's'v'$; *bca* will be the base of the cone in H. Through *m* draw *mc* tangent to *bca*. It will be the horizontal trace of the parallel plane, and HF, parallel to it and tangent to *uvw*, is the required horizontal trace. Let the student construct the vertical trace.

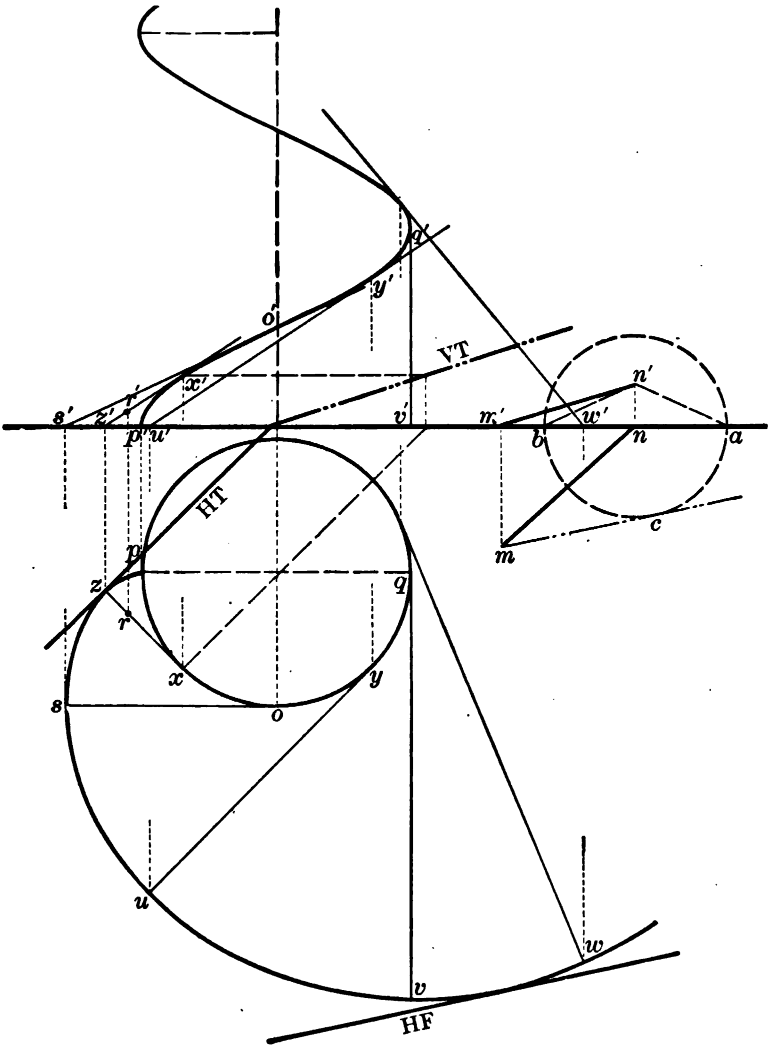


FIG. 64.

128. By an examination of the preceding problems it will be seen:

(1) That, in general, only one plane can be drawn tangent to a single-curved surface at a given point.

(2) That the number which can be drawn through a given point without the surface, and tangent along an element, will be limited.

(3) That the number which can be drawn parallel to a given straight line, and tangent along an element, is also limited.

(4) That in general a plane cannot be passed through a given straight line and tangent to a single-curved surface. If, however, the given line lies on the convex side of the surface, and is parallel to the rectilinear elements of a cylinder, or passes through the vertex of a cone, or is tangent to a line of the surface, the problem is possible.

129. PROBLEM 45. **To find the intersection of a helical convolute by a plane,** intersect the surfaces by a system of auxiliary planes tangent to the projecting cylinder of the helix. These intersect the convolute in rectilinear elements, and the plane in straight lines, the intersection of which will be points of the required curve.

130. PROBLEM 46. **To develop a helical convolute.** When the surface is laid out into a plane, the helix takes the form of a circular arc whose radius $R = r + \frac{P}{4\pi^2 r}$, in which P is the pitch of the helix, and r is the radius of the horizontal projection of the helix. The developed elements of the surface are tangent to the developed helix, and the developed base of the surface is the involute of this curve (Art. 124).

To draw the development of any curve of the surface, the proper distances may be laid off along the elements, either from the points where the elements cut the base, or from the points of tangency of the elements with the developed helix.

Ex. 109. Assume a helix whose H projection is a circle of $1\frac{1}{2}''$ diameter, and whose pitch is $4''$. (*a*) With this as a directrix construct the two projections of a helical convolute, showing eight elements in the first half convolution. (*b*) Find the intersection of this surface with an oblique plane. (*c*) Develop the surface, showing the curve of intersection upon the development. (*d*) Trace the helix upon a wooden cylinder of $1\frac{1}{2}''$ diameter, cut the development from the drawing paper, and fit the inner curve to the helix, thus causing the paper to take the true shape of the convolute.

WARPED SURFACES WITH A PLANE DIRECTER

131. There are a great variety of warped surfaces, differing from one another in their mode of generation and properties.

The most simple are those which may be generated by a straight line *generatrix*, moving so as to touch two other lines as *directrices*, and parallel to a given plane, called a *plane directer.*

Such surfaces are *warped surfaces with two linear directrices and a plane directer.*

They, as all other warped surfaces, may be represented by projecting one or more curves of the surface, and the principal rectilinear elements.

132. PROBLEM 47. **Given the directrices and plane directer of a warped surface, to construct the rectilinear element through a point of either directrix.** Pass a plane through the given point parallel to the plane directer, find the point in which the other directrix pierces this plane, and join it with the given point.

Construction. Let MN and PQ, Fig. 65, be any two linear directrices, T the plane directer, and O any point of the first directrix.

Assume any line of the plane directer, as CD (Art. 27), and through the different points of this line draw straight lines

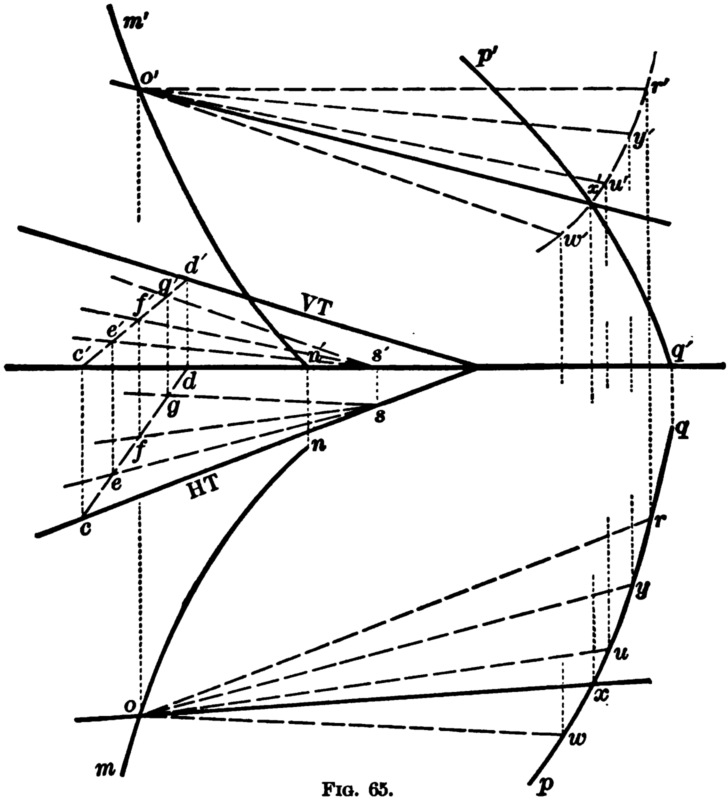


FIG. 65.

SE, SF, etc., to any point, as S, of HT. Through O draw a system of lines, OR, OY, etc., parallel respectively to SE, SF, etc. These will form a plane through O, parallel to the plane T, and pierce the horizontal projecting cylinder of PQ, in the points R, Y, W, etc. These points, being joined, will form the curve RW, which intersects PQ in X, and this is the point in which the auxiliary plane cuts the directrix PQ. OX will then be the required element. If the curve $r'w'$ should intersect $p'q'$ in more than one point, two or more elements passing through O would thus be determined.

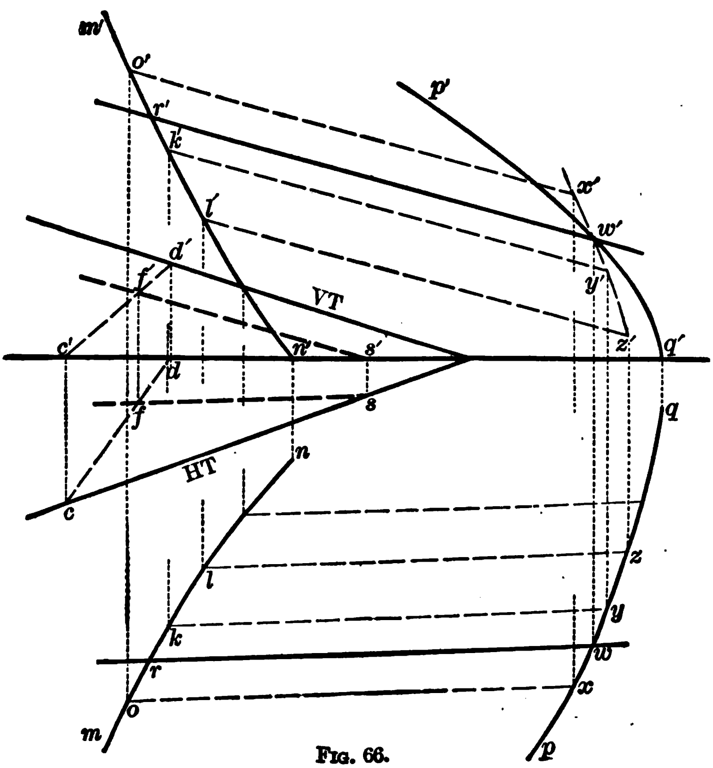


Fig. 66.

133. Problem 48. **Given the directrices and plane directer of a warped surface, to construct an element parallel to a given straight line; this line being in the plane directer, or parallel to it.** Draw through the different points of either directrix, lines parallel to the given line. These form the rectilinear elements of a cylinder, parallel to the given line. If the points in which the second directrix pierces this cylinder be found, and lines be drawn through them parallel to the given line, each will touch both directrices, and be the element required.

Construction. Let the surface be given as in the preceding article and let FS, Fig. 66, be the given line. Through

the points O, K, L, etc., draw OX, KY, LZ, etc., parallel to FS. These lines pierce the horizontal projecting cylinder of the directrix PQ, in the points X, Y, Z, etc., which, being joined, form the curve XY, intersecting PQ in W. Through W draw WR, parallel to FS; it is a required element, and there may be two or more as in the preceding article.

134. Classes of warped surfaces with a plane directer. If both directrices are straight lines, the surface is called a *hyperbolic paraboloid.* If one directrix is a straight line and the other a curved line, the surface is a *conoid.* If both directrices are curved, the surface is a *cylindroid.* The surfaces in Figs. 65 and 66 are cylindroids.

If the rectilinear directrix of a conoid is perpendicular to the plane directer, it is a *right conoid*, and this directrix is the *line of striction.*

The Hyperbolic Paraboloid

135. The **hyperbolic paraboloid** is so named because its intersection by a plane may be proved to be either a hyperbola or a parabola.

Its rectilinear elements may be constructed by the principles in Arts. 132 and 133. In the first case, two straight lines through the given point will determine the auxiliary plane, and the point in which it is pierced by the second directrix may be determined at once, as in Art. 43. In the second case, the cylinder becomes a plane, and the point in which it is pierced by the second directrix is also determined, as in Art. 43.

136. Proposition XXXIV. **The rectilinear elements of a hyperbolic paraboloid divide the directrices proportionally.** For these elements are in a system of planes parallel to the plane directer and to one another, and these planes divide the directrices into proportional parts at the points where they are intersected by the elements.

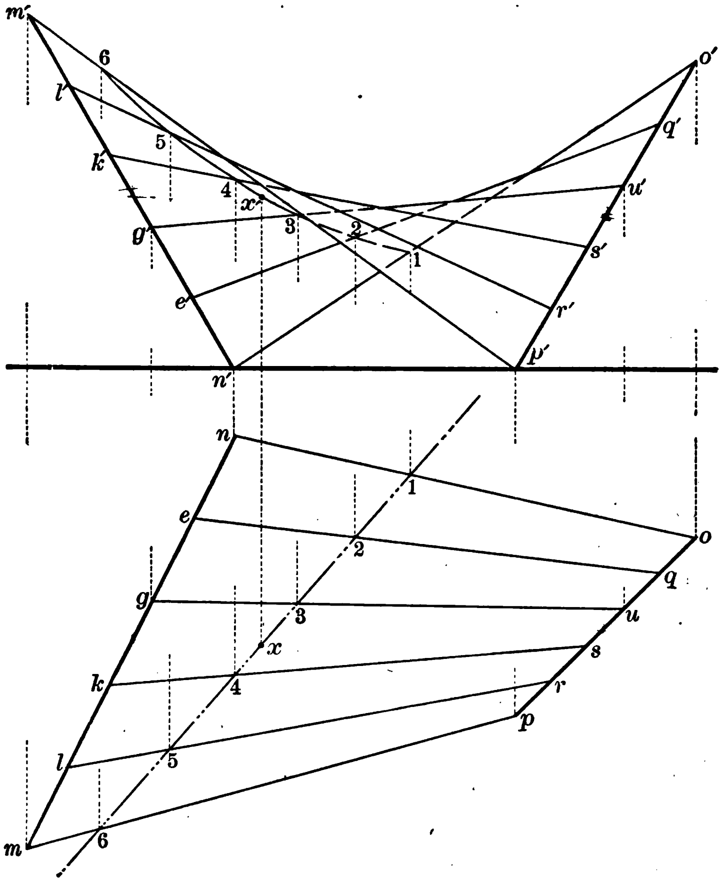


FIG. 67.

137. PROPOSITION XXXV. *Conversely*, — **If two straight lines be divided into any number of proportional parts, the straight lines joining the corresponding points of division will lie in a system of parallel planes, and be elements of a hyperbolic paraboloid, the plane directer of which is parallel to any two of these dividing lines.**

Thus, let MN and OP, Fig. 67, be any two rectilinear direc-

trices. Take any distance, as *ml*, and lay it off on *mn* any number of times, as *ml*, *lk*, *kg*, etc. Take also any distance, as *pr*, and lay it off on *po* any number of times, as *pr*, *rs*, *su*, *qo*, etc. Join the corresponding points of division by straight lines, *lr*, *ks*, *gu*, etc.; these will be the horizontal projections of rectilinear elements of the surface. Through *m*, *l*, *k*, etc., and *p*, *r*, *s*, etc., erect perpendiculars to the ground line, to m', l', k', etc., and p', r', s', etc., and join the corresponding points by the straight lines $l'r'$, $k's'$, $g'u'$, etc.; these will be the vertical projections of the elements.

138. Problem 49. **To assume a point on any warped surface.** We first assume one of its projections, as the horizontal. If this lies on the horizontal projection of an element already drawn, its vertical projection may easily be found on the vertical projection of that element. Otherwise, pass a plane through the point perpendicular to H. It will intersect the rectilinear elements in points which, joined, will give a line of the surface, and the vertical projection of the assumed point will lie in the vertical projection of this line.

In Fig. 67, *x* is the assumed horizontal projection of a point on the surface of the hyperbolic paraboloid. The method of finding the location of x' will be evident from an examination of the figure.

139. Proposition XXXVI. **If any two rectilinear elements of a hyperbolic paraboloid be taken as directrices, with a plane directer parallel to the first directrices, and a surface be thus generated, it will be identical with the first surface.**

To prove this we have only to prove that any point of an element of the *second generation* is also a point of the *first generation.*

Thus, let MN and OP, Fig. 68, be the directrices of the first generation, and NO and MP any two rectilinear elements. Through M draw MW parallel to NO. The plane WMP will

be parallel to the plane directer of the first generation, and may be taken for it.

Let NO and MP be taken as the new directrices, and let ST be an element of the second generation, the plane directer being parallel to MN and OP. Through U, any point of ST, pass a plane parallel to WMP, cutting the directrices MN and OP in Q and R. Join QR; it will be an element of the first

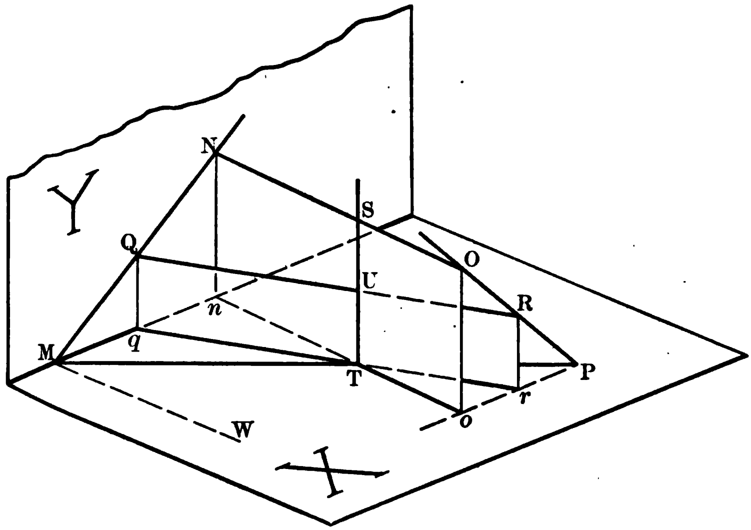


FIG. 68.

generation (Art. 132). Draw N*n* and Q*q* parallel to ST, piercing the plane WMP in *n* and *q*. Also draw O*o* and R*r* parallel to ST, piercing WMP in *o* and *r*; and draw *no*, intersecting MP in T, since N*n*, ST, and O*o* are in the same plane. M, *q*, and *n* will be in the same straight line, as also P, *r*, and *o*; and since MN and N*n*, intersecting at N, are parallel to the plane directer of the second generation, their plane will be parallel to it, as also the plane of PO and O*o*; hence, these planes being parallel, their intersections, M*n* and P*o*, with the plane WMP, will be parallel. Draw *qr*.

Since Qq and Nn are parallel, we have

$$MQ : QN :: Mq : qn.$$

Also $$PR : RO :: Pr : ro.$$

But (Art. 136), $$MQ : QN :: PR : RO;$$

hence, $$Mq : qn :: Pr : ro;$$

and the straight line, qr, must pass through T, and the plane of the three parallels, Qq, ST, and Rr, contains the element QR, which must therefore intersect ST at U. Hence, any point of a rectilinear element of the second generation is also a point of an element of the first generation, and the two surfaces are identical. It follows from this, that **through any point of a hyperbolic paraboloid two rectilinear elements can always be drawn.**

Planes Tangent to Warped Surfaces with Plane Directer. Intersections

140. Proposition XXXVII. **A plane tangent to a warped surface, although it contains the rectilinear element passing through the point of contact, cannot contain its consecutive element, and therefore can, in general, be tangent at no other point of the element.**

141. Proposition XXXVIII. **If a plane contain a rectilinear element of a warped surface, and be not parallel to the other elements, it will be tangent to the surface at some point of this element.** For this plane will intersect each of the other rectilinear elements in a point; and these points being joined, will form a line which will intersect the given element. If at the point of intersection a tangent be drawn to this line, it will lie in the tangent plane (Art. 86). The given element, being its own tangent (Art. 60), also lies in the tangent plane. The

plane of these two tangents, that is, the intersecting plane, is therefore tangent to the surface at this point (Art. 86). It is thus seen that, in general, a plane tangent to a warped surface is also an *intersecting plane.*

If the intersecting plane be parallel to the rectilinear elements, there will be no curve of intersection formed as above, and the plane will not be tangent.

142. Since a plane tangent to a warped surface must contain the rectilinear element passing through the point of contact (Art. 88), we can at once determine one line of the tangent plane. A second line may then be determined in accordance with the rule in Art. 86.

When the surface has two different generations by straight lines, the plane of the two rectilinear elements passing through the given point will be the required plane.

143. PROBLEM 50. **To pass a plane tangent to a hyperbolic paraboloid at a given point of the surface.**

Let MN and PQ, Fig. 69, be the directrices, and MP and NQ any two elements of the surface, and let O, assumed as in Art. 138, be the given point.

Analysis. Since through the given point a rectilinear element of each generation can be drawn (Art. 139), we have simply to construct these two elements and pass a plane through them (Art. 142).

Construction. Through O draw OE and OF parallel respectively to NQ and MP. These will determine a plane parallel to the plane directer of the first generation (Art. 135). This plane cuts the directrix PQ in the point U (Art. 43). Join U with O, and we have an element of the first generation (Art. 132). Let NQ and MP be taken as directrices of the second generation. Through O draw OC and OD parallel respectively to MN and PQ. They will determine a plane parallel to the plane directer of the second generation (Art. 135).

This plane cuts MP in W, and OW will be an element of the second generation, and the tangent plane is determined as in Art. 30.

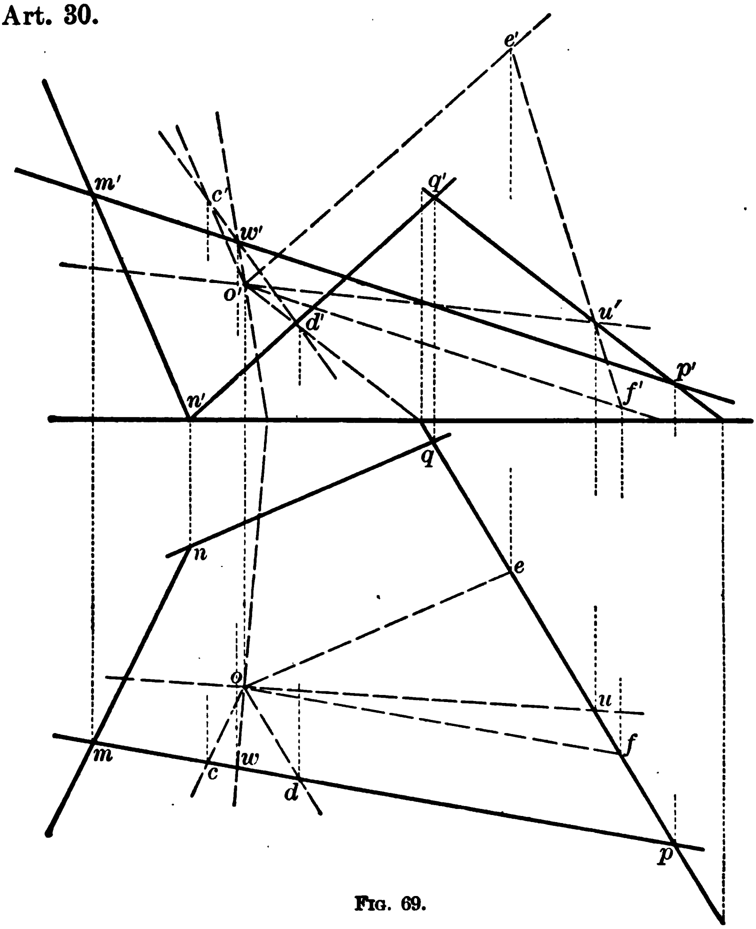


FIG. 69.

144. Two curved surfaces are tangent to each other when they have at least one point in common, through which if any intersecting plane be passed, the lines cut from the surfaces will be tangent to each other at this point. This will evidently be the case when they have a common tangent plane at this point.

145. PROPOSITION XXXIX. **If two single-curved surfaces are tangent to each other at a point of a common element, they will be tangent all along this element.** For the common tangent plane will contain this element and be tangent to each surface at every point of the element (Art. 89).

This principle is not true with warped surfaces.

146. PROPOSITION XL. **But if two warped surfaces, having two directrices, have a common plane directer, a common rectilinear element, and two common tangent planes, the points of contact being on the common element, they will be tangent all along this element.** For if through each of the points of contact any intersecting plane be passed, it will intersect the surfaces in two lines, which will have, besides the given point of contact, a second consecutive point in common (Art. 86). If, now, the common element be moved upon the lines cut from either surface, as directrices, and parallel to the common plane directer, into its consecutive position, containing these second consecutive points, it will evidently lie in both surfaces, and the two surfaces will thus contain two consecutive rectilinear elements. If, now, *any plane* be passed, intersecting these elements, two lines will be cut from the surfaces, having two consecutive points in common, and therefore tangent to each other; hence the surfaces will be tangent all along the common element (Art. 144).

147. PROBLEM 51. **To pass a plane tangent to a conoid or a cylindroid at a given point of the surface.** Let MN and PQ, Fig. 70, be the two directrices of a cylindroid having V for its plane directer, and let O be the given point. At the points X and Y, in which the rectilinear element through O intersects the directrices, draw a tangent to each directrix, as XZ and YU. On these tangents as directrices move XY parallel to V. It will generate a hyperbolic paraboloid (Art. 135) having with the given surface the common element XY, and a common

tangent plane at each of the points X and Y, since the plane of the two lines XY and XZ, and also that of XY and YU is tangent to both surfaces (Art. 86). The two surfaces are therefore tangent all along the common element XY (Art. 146). If, then, at O we pass a plane tangent to the hyperbolic paraboloid, it will be also tangent to the given surface at the same point. Let the student complete the construction.

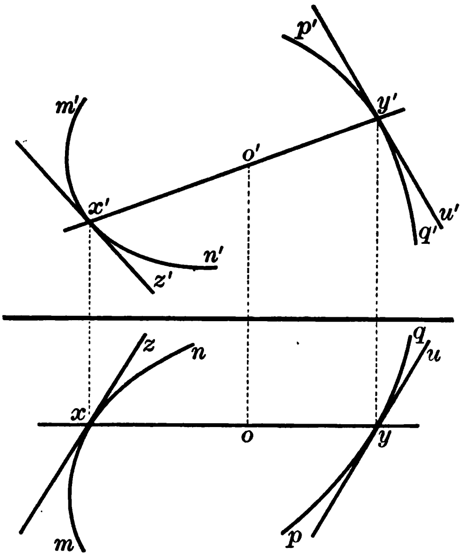


Fig. 70.

148. Problem 52. **To pass a plane through a given straight line, and tangent to a warped surface,** it is necessary only to produce the line until it pierces the surface, and through the point thus determined draw the rectilinear element of the surface. This, with the given line, will determine a plane tangent to the surface at some point of the element (Art. 141).

If there be two rectilinear elements passing through this point, each will give a tangent plane, and the number of tangent planes will depend upon the number of points in which the line pierces the surface. If the given line be parallel to a rectilinear element, it pierces the surface at an infinite distance, and the tangent plane will be determined by the two parallel lines.

149. Problem 53. **To find the intersection of any warped surface with a plane directer, by an oblique plane,** intersect

by planes parallel to the plane directer. Each will cut from the surface one or more rectilinear elements, and from the plane a straight line, the intersection of which will be points of the required curve.

Or otherwise, intersect by planes perpendicular to either plane of projection; cutting elements from the surface, and straight lines from the plane.

Ex. 110. Assume two non-parallel, non-intersecting lines, AB and CD, as the directrices of a hyperbolic paraboloid, and a plane T as directer. Determine two elements of the surface by Art. 132. Then determine three other elements by Art. 137. Assume a point X of the surface not on any of these five elements (Art. 138). Construct an element MN through X (Art. 143). Construct an element OP of the second generation through X. Pass a plane S tangent to the surface at X.

Ex. 111. Find the intersection between a hyperbolic paraboloid similar to the one in Fig. 67, and an oblique plane (Art. 149).

Ex. 112. Assume a helix as the curvilinear directrix of a right conoid, and the axis of the helix as its rectilinear directrix, perpendicular to H, the horizontal plane being the plane directer. Then assume an oblique cone with base in H, and vertex in the axis of the helix. Find the intersection between the cone and conoid by passing a system of auxiliary planes in such a manner as to cut elements from both surfaces.

Ex. 113. Assume a circle in H, and a straight line parallel to the ground line as the directrices of a right conoid, with the profile plane as directer. Assume a straight line FG in such a way as to cut the surface of the conoid. Pass a plane T through FG and tangent to the surface (Art. 148). Then find the point of tangency (Art. 141). Analyze.

Ex. 114. Assume a cylindroid similar to that of Fig. 66, and any straight line in space not perpendicular to the plane

directer. Pass a plane S tangent to the cylindroid and perpendicular to this line. The solution depends upon Arts. 133 and 148. Let the student analyze the problem.

Ex. 115. Assume a conoid with H as the plane directer, and pass a plane tangent to the surface at an assumed point of the surface (Art. 147). To facilitate the construction, let the projections of the curvilinear directrix be circular arcs.

The Helicoid

150. A **helicoid** is a warped surface generated by a straight line so moving that it constantly touches a helix (Art. 70), and the axis of the helix, and makes a constant angle with the axis.*

The surface may be represented by taking the axis perpendicular to the horizontal plane, Fig. 71, o being its horizontal, and $o'n'$ its vertical projection; and constructing the helix PRQ (Art. 71), and the projections of the several elements.

151. To construct the projections of the elements of the surface. Let P be the point in H from which the helix springs, and PO, parallel to V, the first position of the generatrix; the angle $p'o'n'$ being the constant angle between the elements and the axis. To construct the projections of an element through the point X on the helix, we regard the generatrix as having moved along the helix from P to X. In so doing it has risen through a vertical distance equal to $s'x'$. The point where the generatrix intersects the axis will have risen through the same distance. Hence we lay off $o'o'_1$ equal to $s'x'$; and $x'o'_1$ will be the vertical projection of the element, and xo its horizontal. In the same way the element (oy, o'_2y') may be assumed.

* For a discussion of the general case of a "helicoidal surface," see MacCord's Descriptive Geometry.

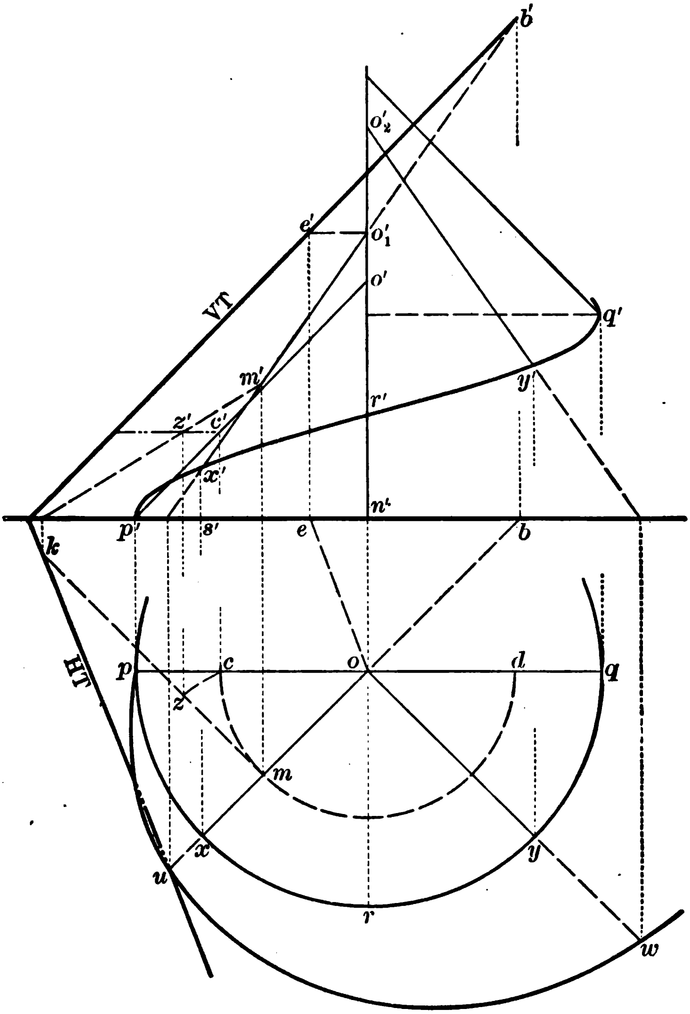


FIG. 71.

152. To assume any point on the surface, we first assume its horizontal projection, as m, and construct the two projections of the element, as (ox, o'_1x'). The vertical projection of

the point will then be found on the vertical projection of the element.

If the *vertical* projection of the point is first assumed, and if it does not lie on an element already drawn, it will be necessary to follow the method of Art. 138.

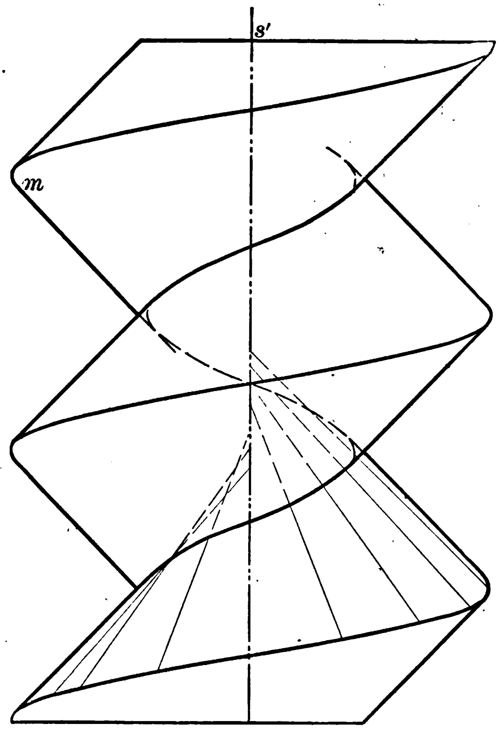


FIG. 72.—Screw-thread. Helicoidal Surface.

153. To construct the horizontal trace of the surface, we find the horizontal piercing points, *p*, *u*, *w*, etc., of the several elements, and the curve *puw* is the horizontal trace of the surface. This curve is the *Spiral of Archimedes.*

The surface is manifestly a warped surface, since, from the nature of its generation, no two consecutive positions of the generatrix can be parallel or intersect.

This surface is an important one, as it forms the curved surface of the thread of the ordinary screw (Fig. 72).

If the generatrix is perpendicular to the axis, the helicoid becomes a particular case of the right conoid (Art. 134), the horizontal plane being the plane directer, and any helix the curvilinear directrix. This is the surface in the square-

threaded screw. The joints in certain types of skew arches are also of this form.

154. Problem 54. **To pass a plane tangent to a helicoid at a point of the surface.**

Let the surface be given as in Fig. 71 and let M be the given point.

Analysis. The tangent plane must contain the rectilinear element passing through the given point, and also the tangent to the helix at this point (Art. 86). The plane of these two lines will then be the required plane.

Construction. MX is the rectilinear element through M. It pierces H at *u*; *cmd* is the horizontal projection of the helix through M. Draw the tangent to this helix at M, as in Art. 73. *mz* will be its horizontal projection, Z the point in which it pierces the horizontal plane through C, and $m'z'$ its vertical projection. This tangent pierces H at *k*; hence *ku* is the horizontal trace of the required plane, and VT is the vertical trace.

155. Problem 55. **To pass a plane tangent to a helicoid and perpendicular to a given straight line.**

Analysis. If, with any point of the axis as a vertex, we construct a cone whose rectilinear elements shall make with the horizontal plane the same angle as that made by the rectilinear elements of the given surface, and through the vertex of this cone pass a plane perpendicular to the given line (Art. 44), it will, if the problem be possible, cut from the cone two elements, each of which will be parallel to a rectilinear element of the helicoid and have the same horizontal projection. If through either of these elements of the helicoid a plane be passed parallel to the auxiliary plane, it will be tangent to the surface (Art. 141) and perpendicular to the given line.

Let the problem be constructed in accordance with the analysis.

156. PROBLEM 56. **To find the intersection of a helicoid by a plane.** Let the surface be given as in Art. 150. Intersect by a system of auxiliary planes through the axis. These will cut from the surface rectilinear elements, and from the plane straight lines, the intersection of which will be points of the required curve.

Let the construction be made, and the curve and its tangent represented in true dimensions.

Ex. 116. To find the point in which a given line pierces a helicoid. Analyze and construct.

Ex. 117. Assume the vertical projection of a point on a helicoid, and find its horizontal projection (Art. 138).

Ex. 118. Pass a plane through a given straight line and tangent to a helicoid, and find the point of tangency (Art. 141). When is the problem impossible?

Ex. 119. Find the intersection between a helicoid and a cylinder whose elements are parallel to the axis of the helicoid. Analyze.

Ex. 120. Find the intersection between a helicoid and an oblique cone—axis of helicoid perpendicular to H, vertex of cone in axis of helicoid, and base of cone lying in V. Analyze.

WARPED SURFACES WITH THREE LINEAR DIRECTRICES

157. A third class of warped surfaces consists of those which may be generated by moving a straight line so as to touch three lines as directrices, or *warped surfaces with three linear directrices.*

To construct a rectilinear element of this class of surfaces passing through a given point on one of the directrices, draw through this point a system of straight lines intersecting either of the other directrices; these form the surface of a cone which will be pierced by the third directrix in one or more

points, through which and the given point straight lines may be drawn that will touch the three directrices, and be required elements.

Construction. Let MN, OP, and QR, Fig. 73, be any three linear directrices, and M a given point on the first.

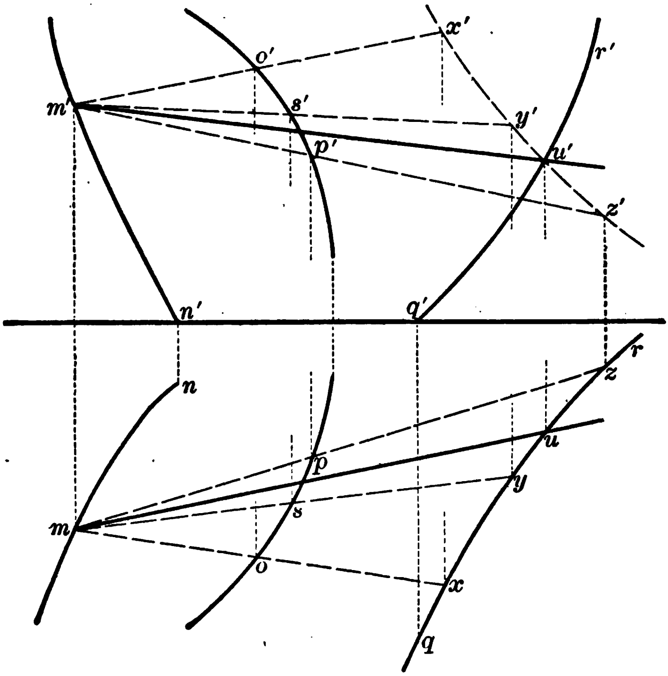


FIG. 73.

Through M draw the lines MO, MS, MP, etc., intersecting OP in O, S, P, etc. They pierce the horizontal projecting cylinder of QR in X, Y, Z, etc., forming a curve, XYZ, which intersects QR in U, the point in which QR pierces the surface of the auxiliary cone. MU is then a required element, two or more of which would be determined if XZ should intersect QR in more than one point.

158. Classes of warped surfaces with three linear directrices. The directrices may all be curved lines, or two of them may be curved and one straight, or one may be curved and two straight, or all three may be straight. The last-named surface is the simplest and most common, and is called a *hyperboloid of one nappe*, as many of its intersections by planes are hyperbolas, while the surface itself is unbroken. For applications of the first two of the above-named surfaces, see the discussion of the gate-recess and the cow's-horn arch in any work on Stereotomy.

159. To construct a rectilinear element of a hyperboloid of one nappe, the method of Fig. 73 is here much simplified, as the auxiliary cone becomes a plane; and the point in which this plane is pierced by the third directrix is found as in Art. 42 or 43.

160. Proposition XLI. **If two warped surfaces, having three directrices, have a common element and three common tangent planes, the points of contact being on this element, they will be tangent all along this element.** For if through each point of contact any intersecting plane be passed, it will intersect the surfaces in two lines, which, besides the given point of contact, will have a second consecutive point in common. If the common element be moved upon the three lines cut from either surface as directrices, to its consecutive position, so as to contain the second consecutive points, it will evidently lie in both surfaces; hence the two surfaces contain two consecutive rectilinear elements and will be tangent all along the common element.

161. To pass a plane tangent to a warped surface with three curvilinear directrices at a point on the surface, a tangent may be drawn to each at the point in which the rectilinear element through the given point intersects it; and then this element may be moved on these three tangents as directrices, generating a hyperboloid of one nappe (Art. 158), which will be tangent

to the given surface all along a common element (Art. 160). A plane tangent to this auxiliary surface at the given point will also be tangent to the given surface.

162. An infinite number of planes may, in general, be passed through a point without a warped surface, and tangent to it. For if, through the point, a system of planes be passed intersecting the surface, tangents may be drawn from the point to the curves of intersection, and these will form the surface of a cone tangent to the warped surface. Any plane tangent to this cone will be tangent to the warped surface, and pass through the point.

Also, an infinite number of planes may, in general, be passed tangent to a warped surface and parallel to a straight line. For if the surface be intersected by a system of planes parallel to the line, and tangents parallel to the line be drawn to the sections, they will form the surface of a cylinder tangent to the warped surface. Any plane tangent to this cylinder will be tangent to the warped surface and parallel to the given line.

163. Other varieties of warped surfaces may be generated by moving a straight line so as to touch two lines, having its different positions, in succession, parallel to the different rectilinear elements of a cone; or by moving it parallel to a given plane, so as to touch two surfaces, or one surface and a line; or so as to touch three surfaces, two surfaces and a line, one surface and two lines; or in general, so as to fulfill *any three reasonable conditions.*

It should be remarked that in all these cases the directrices should be so chosen that the surface generated will be neither a plane nor a single-curved surface. Also, that these surfaces, as all others, may be generated by curves moved in accordance with a law peculiar to each variety.

164. If a curve be moved in any way so as not to generate either a plane, a single-curved surface, or a warped surface, it

will generate a *double-curved surface*, the simplest variety of which is the *surface of a sphere.*

For a general classification of surfaces, the student is referred to the arrangement on page 76.

SURFACES OF REVOLUTION

165. A surface of revolution is a surface which may be generated by revolving a line about a straight line as an axis (Art. 34).

From the nature of this generation it is evident that any intersection of such a surface by a plane perpendicular to the axis is the circumference of a circle.

Meridian line and meridian plane. If the surface be intersected by a plane passing through the axis, the line of intersection is a *meridian line*, and the plane a *meridian plane;* and it is also evident that all meridian lines of the same surface are equal, and that the surface may be generated by revolving any one of these meridian lines about the axis.

166. PROPOSITION XLII. **If two surfaces of revolution having a common axis intersect, the line of intersection must be the circumference of a circle whose plane is perpendicular to the axis and whose center is in the axis.** For if a plane be passed through any point of the intersection and the common axis, it will cut from each surface a meridian line (Art. 165), and these meridian lines will have the point in common. If these lines be revolved about the common axis, each will generate the surface to which it belongs, while the common point will generate the circumference of a circle common to the two surfaces, and therefore their intersection. Should the meridian lines intersect in more than one point, the surfaces will intersect in two or more circumferences.

167. The cylinder of revolution. The simplest curved surface of revolution is that which may be generated by a straight

line revolving about another straight line to which it is parallel. This is evidently a cylindrical surface (Art. 79), and if the plane of the base be perpendicular to the axis, it is a *right cylinder with a circular base.*

The cone of revolution. If a straight line be revolved about another straight line which it intersects, it will generate a conical surface (Art. 83), which is evidently a *right cone*, the axis being the line with which the rectilinear elements make equal angles.

These are the only two single-curved surfaces of revolution.

The Hyperboloid of Revolution of One Nappe

168. If a straight line be revolved about another straight line not in the same plane with it, it will generate a surface of revolution called a *hyperboloid of revolution of one nappe.*

To prove that this is a warped surface, let us take the horizontal plane perpendicular to the axis, and the vertical plane parallel to the generatrix in its first position, and let c, Fig. 74, be the horizontal, and $c's'$ the vertical projection of the axis, and MP the generatrix; cm will be the horizontal, and $c'm'$ the vertical projection of the shortest distance between these two lines (Art. 53).

As MP revolves about the axis, CM will remain perpendicular to it, and M will describe a circumference which is horizontally projected in mxy, and vertically in $y'x'$; and as CM is horizontal, its horizontal projection will be perpendicular to the horizontal projection of MP in all of its positions (Prop. XX, Art. 14), and remain of the same length; hence the horizontal projection of MP in any position will be tangent to the circle mxy. No two consecutive positions can therefore be parallel. Neither can they intersect; for from the nature of the motion any two must be separated at any point by the elementary arc of the circle described by that point. *The surface is therefore a warped surface.*

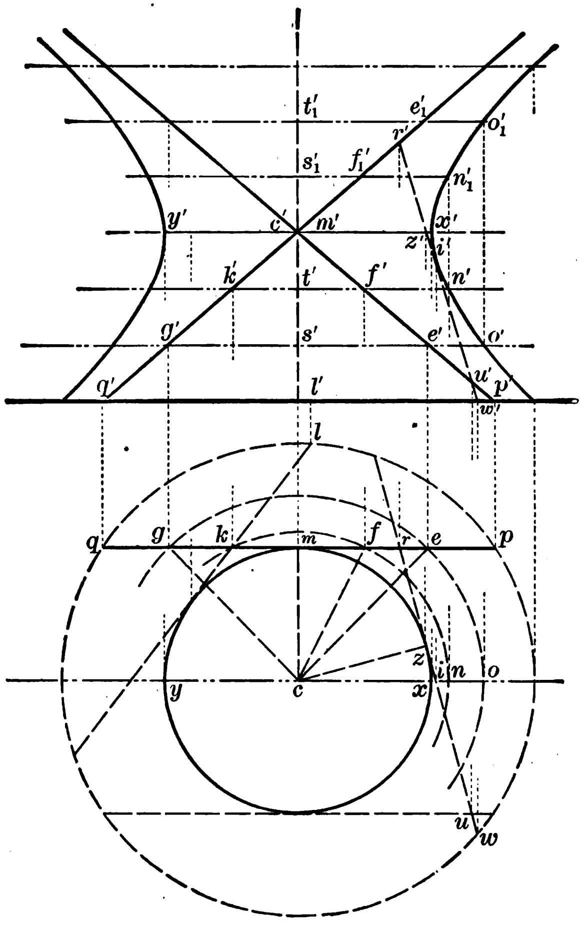


FIG. 74.

The point P, in which MP pierces H, generates the circle *pwq*, which may be regarded as the *base* of the surface.

The circle generated by M is the smallest circle of the surface and is the *circle of the gorge*.

169. To assume a rectilinear element, we take any point in the base, *pwq*, and through it draw a tangent to *xmy*, as *wz* ;

this will be the horizontal projection of an element. Through z erect the perpendicular zz'; z' will be the vertical projection of the point in which the element crosses the circle of the gorge, and $w'z'$ will be the vertical projection of the element.

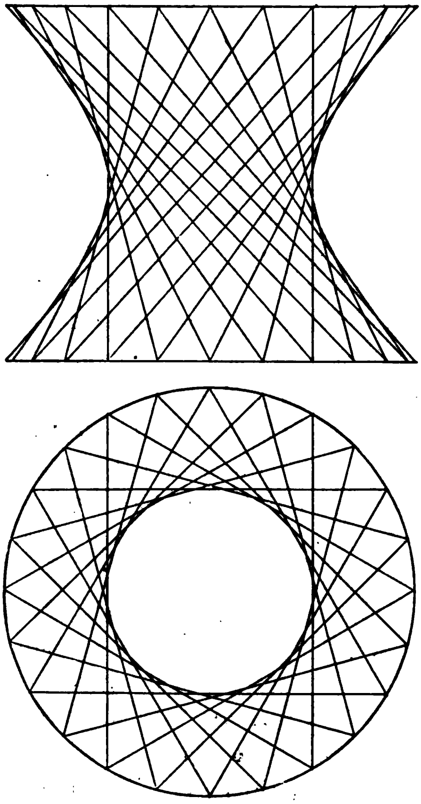


FIG. 75.—Hyperboloid of Revolution of one Nappe, showing Double Generation.

To assume a point of the surface, we may first assume its horizontal projection, as u, and through it draw the horizontal projection of an element. The vertical projection of the point will be found on the vertical projection of the element.

170. Double generation. If through the point M a second straight line as MQ be drawn parallel to the vertical plane, and making with the horizontal plane the same angle as MP, and this line be revolved about the same axis, it will generate the same surface. For if any plane be passed perpendicular to the axis, as the plane whose vertical trace is $e'g'$, it will cut MP and MQ in two points E and G, equally distant from the axis, and these points will, in the revolution of MP and MQ, generate the same circumference; hence the two surfaces must be identical. The surface having two generations by different straight lines,

it follows that *through any point of the surface two rectilinear elements can be drawn.*

171. Proof that this surface is a hyperboloid of one nappe. Since the points E and G generate the same circumference, it follows that as MP revolves, MQ remaining fixed, the point E will, at some time of its motion, coincide with G, and the generatrix, MP, intersect MQ in G. In the same way any other point, as F, will come into the point K of MQ, giving another element of the first generation, intersecting MQ at K; and so for each of the points of MP in succession. In this case *kl* will be the horizontal projection of the element of the first generation. Hence, *if the generatrix of either generation remains fixed, it will intersect all the elements of the other generation.*

If, then, any three elements of either generation be taken as directrices, and an element of the other generation be moved so as to touch them, it will generate the surface. It is therefore a *hyperboloid of one nappe* (Art. 158).

A hyperboloid of revolution of one nappe may thus be generated by *moving a straight line so as to touch three other straight lines*, equally distant from a fourth (the axis), the distances being measured in the same plane perpendicular to the fourth, and the directrices making equal angles with this plane.

172. To represent the surface when the first position of the generatrix is not parallel to V, the circle of the gorge may be constructed thus: Let WZ be the first position of the generatrix, and through *c* draw *cz* perpendicular to *wz*: it will be the horizontal projection of the radius of the required circle. The point, of which *z* is the horizontal projection, is vertically projected at z', and *mzx* will be the horizontal, and $y'z'x'$ the vertical projection of the circle of the gorge. With *c* as a center and *cw* as a radius (*w* being the horizontal piercing point of the line), describe the base *pwq*, and the surface will be fully represented.

173. To construct a meridian curve of this surface, we pass a plane through the axis parallel to the vertical plane. It will intersect the horizontal circles generated by the different points of the generatrix in points of the required curve, which will be vertically projected into a curve equal to itself (Art. 58).

Thus the horizontal plane whose vertical trace is $e'g'$ intersects the generatrix in E, and eo is the horizontal projection of the circle generated by this point, and O is the point in which this circle pierces the meridian plane. In the same way the points whose vertical projections are n', x', n'_1, o'_1, etc., are determined.

174. Proof that the meridian curve is a hyperbola. The plane whose vertical trace is $e'_1 o'_1$ at the same distance from $y'x'$ as $e'o'$, evidently determines a point o'_1, at the same distance from $y'x'$ as o'; hence the chord $o'o'_1$ is bisected by $y'x'$, and the curve $o'x'o'_1$ is symmetrical with the line $y'x'$.

The distance $e'o'$, equal to $ce - me$, is the difference between the hypotenuse ce and base me of a right-angled triangle having the altitude mc. As the point o' is further removed from x', the altitude of the corresponding triangle remains the same, while the hypotenuse and base both increase. If we denote the altitude by a, and the base and hypotenuse by b and h respectively, we have

$$h^2 - b^2 = a^2, \text{ and } h - b = \frac{a^2}{h+b},$$

from which it is evident that the difference $e'o'$ continually diminishes as the point o' recedes from x'; that is, the curve $x'n'o'$ continually approaches the lines $p'm'$ and $q'm'$, and will touch them at an infinite distance. These lines are then asymptotes to the curve (Art. 69).

If, now, any element of the first generation, as the one passing through I, be drawn, it will intersect the element of the second generation MQ in R, and the corresponding element on the opposite side of the circle of the gorge in U.

Since this element has but one point in common with the meridian curve, and no point of it or of the surface can be vertically projected on the right of this curve, the vertical projection $u'r'$ must be tangent to the curve at i'. But since ri is equal to iu, $r'i'$ must be equal to $i'u'$, or the part of the tangent included between the asymptotes *is bisected at the point of contact.* This is a property peculiar to the hyperbola, and the meridian curve is therefore a hyperbola; YX being its transverse axis, and the axis of the surface its conjugate (Art. 69). If this hyperbola be revolved about its conjugate axis, it will generate the surface (Art. 165); hence its name.

This is the only warped surface of revolution.

175. PROPOSITION XLIII. **A plane tangent to a surface of revolution is perpendicular to the meridian plane passing through the point of contact.** For this tangent plane contains the tangent to the circle of the surface at this point (Art. 86), and this tangent is perpendicular to the radius of this circle, and also to a line drawn through the point parallel to the axis, since it lies in a plane perpendicular to the axis; and therefore it is perpendicular to the plane of these two lines, which is the meridian plane.

176. PROBLEM 57. **To pass a plane tangent to a hyperboloid of revolution of one nappe at a given point of the surface.**

Let MP, Fig. 76, be one position of the generatrix, and o the horizontal projection of the given point on the surface.

Analysis. Since through the given point two rectilinear elements can be drawn (Art. 170), we have simply to construct these two elements, and pass a plane through them (Art. 142).

Construction. The point M is the nearest to the axis. It therefore generates the gorge circle, horizontally projected in mzs, and vertically projected in a straight line through m' parallel to the ground line (Art. 172). The horizontal pierc-

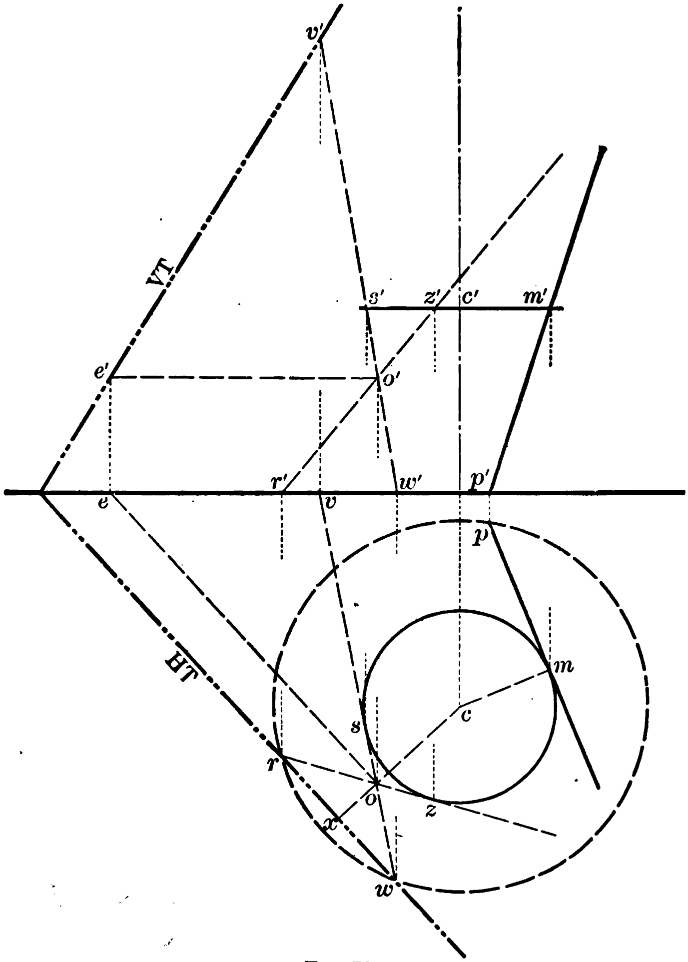


FIG. 76.

ing point p generates the circle of the base. Now from the nature of the surface it is evident that o is the horizontal projection of two points of the surface — one below, and one above the gorge. In this problem we will consider O to be the point *below* the gorge. If we revolve MP about the axis

until it contains the point O, it will take the position *ws*, *w's'*. This is then the element of the first generation through O (Art. 169), and *zr*, *z'r'* will be the element of the second generation (Art. 170). These elements pierce H at *w* and *r*, and *wr* is the horizontal trace of the required plane, and *e'v'* its vertical trace.

Under the assumption that the point O is *above* the gorge, where will the elements of the first and second generations pierce H? Let the student construct the vertical projections of the elements and the traces of the tangent plane.

Since the meridian plane through O must be perpendicular to the tangent plane (Art. 175), its trace *cx* must be perpendicular to *wr*.

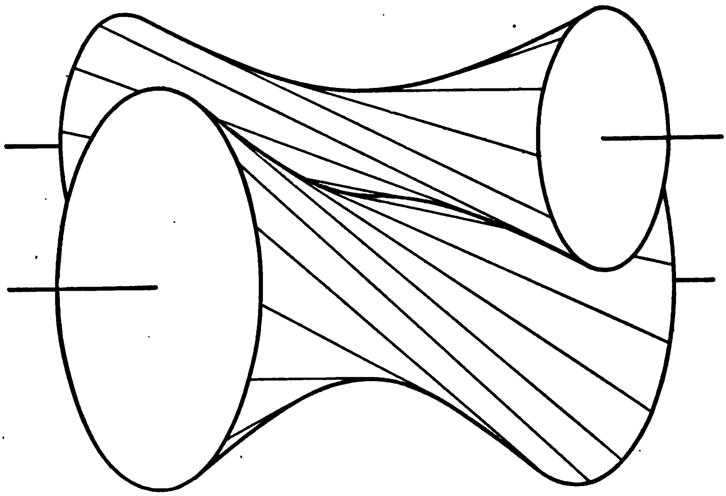


FIG. 77.—Hyperboloids of Revolution Tangent along a Rectilinear Element.

The hyperboloid of revolution of one nappe is, for the sake of brevity, sometimes called the *hypoid*. It is of importance in connection with the design and construction of skew bevel gears. For a further discussion of the properties of this sur-

face, see Rankine's Machinery and Millwork, Arts. 84 and 106, and MacCord's Kinematics of Machinery, Arts. 151–170, 378 and 379.

Ex. 121. Given the axis of a hyperboloid of revolution of one nappe, perpendicular to H, and any position of the generating line,

(*a*) Construct the horizontal trace of the surface.

(*b*) Construct the projections of the gorge circle.

(*c*) Construct the element MN through any point, X, of the surface whose horizontal projection is given, it being assumed that the point is *above* the gorge circle.

(*d*) Construct the element OP through any point, Z, of the surface, whose vertical projection is given, it being assumed that the point is on that part of the surface nearer V.

(*e*) Construct the element RS, of the second generation, through X or Z.

(*f*) Construct the vertical projection of the meridian curve parallel to V.

(*g*) Pass a plane, T, tangent to the surface at the point X or Z.

(*h*) Assume a straight line, EF, and find the point, D, in which the surface is pierced by the line.

(*i*) Pass a plane S through the line EF and tangent to the surface.

(*j*) Pass a plane U tangent to the surface and parallel to an assumed plane W, and find the point of tangency, C. (For a hint, see Art. 155.)

(*k*) Construct the curve of intersection with an oblique plane, a cylinder, cone, convolute, any warped surface with plane directer, any warped surface with three linear directrices, or a helicoid.

(*l*) Construct a line tangent to the curve of intersection at an assumed point on the curve.

Double-Curved Surfaces of Revolution

177. The most simple double-curved surfaces of revolution are:

I. A *spherical surface*, or *sphere*, which may be generated by revolving the circumference of a circle about its diameter.

II. An *ellipsoid of revolution*, or *spheroid*, which may be generated by revolving an ellipse about either axis. When the ellipse is revolved about the transverse axis, the surface is a *prolate spheroid;* when about the conjugate axis, an *oblate spheroid.*

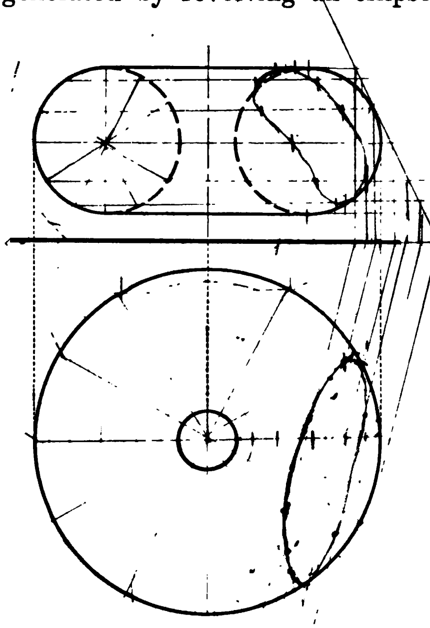


Fig. 78. — Torus.

III. A *paraboloid of revolution*, which may be generated by revolving a parabola about its axis.

IV. A *hyperboloid of revolution of two nappes*, which may be generated by revolving a hyperbola about its transverse axis.

V. A *torus*, which may be generated by revolving a circumference about a line in the plane of and outside the circumference.

178. These surfaces of revolution are usually represented by taking the horizontal plane perpendicular to the axis, and then drawing the intersection of the surface with the horizontal plane; or by taking the horizontal projection of the greatest horizontal circle of the surface for the horizontal projection, and then projecting on the vertical plane the

meridian line which is parallel to that plane for the vertical projection.

179. To assume a point on a surface of revolution, we first assume either projection as the horizontal, and erect a perpendicular to the horizontal plane. Through this perpendicular pass a meridian plane; it will cut from the surface a meridian line which will intersect the perpendicular in the required point or points.

Construction. Let the surface be a prolate spheroid, Fig. 79, and let c be the horizontal, and $c'd'$ the vertical projection of the axis, mon the horizontal projection of its largest circle, and $d'm'c'n'$ the vertical projection of the meridian curve parallel to the vertical plane, and let p be the assumed horizontal projection; p will be the horizontal, and $s'p'$ the vertical projection of the perpendicular to H; cp will be the horizontal trace of the auxiliary meridian plane. If this plane be now revolved about the axis until it becomes parallel to the vertical plane, p will describe the arc pp_1, and p_1 will be the horizontal, and $p'_1q'_1$ the vertical projection of the revolved position of the perpendicular. The meridian curve, in its revolved position, will be vertically projected into its equal $d'm'c'n'$, and p'_1 and q'_1 will be the vertical projections of the two points of intersection in their revolved position. When the meridian plane is revolved to its primitive position, these points will describe the arcs of horizontal circles, projected on H in p_1p, and on V in p'_1p' and q'_1q', and p' and q' will be the vertical projections of the two points of the surface horizontally projected in p.

Let the student assume the vertical projection of a point on the surface, and find its horizontal projection.

Also let the student represent a point on the surface of the torus, Fig. 78, taking the horizontal projection about an eighth of an inch from the inner circle.

Problems relating to Planes Tangent to Surfaces of Revolution

180. These problems are, in general, solved either by a direct application of the rule in Art. 86, taking care to intersect the surface by planes, so as to obtain the two simplest curves of the surface intersecting at the point of contact, or by means of more simple auxiliary surfaces tangent to the given surface.

181. Problem 58. **To pass a plane tangent to a sphere at a given point.**

Analysis. Since the radius of the sphere, drawn to the point of contact, is perpendicular to the tangent to any great circle at this point, and since these tangents all lie in the tangent plane (Art. 86), *this radius must be perpendicular to the tangent plane.* We have then simply to pass a plane perpendicular to this radius at the given point, and it will be the required plane.

Construction. Let the point be assumed as in Art. 179, and the plane passed as in Art. 44. The construction is left for the student.

182. Problem 59. **To pass a plane tangent to any surface of revolution at a given point.**

Let the surface be given as in Art. 179, Fig. 79, and let P be the point.

Analysis. If at the given point we draw a tangent to the meridian curve of the surface, and a second tangent to the circle of the surface at this point, the plane of these two lines will be the required plane (Art. 86).

Construction. Through P pass a meridian plane; *cp* will be its horizontal trace. Revolve this plane about the axis of the surface until it is parallel to V. P_1 will be the revolved position of the point of contact. At p'_1 draw p'_1y' tangent to $c'n'd'm'$ (Art. 67). It will be the vertical projection of the revolved position of a tangent to the meridian curve at P.

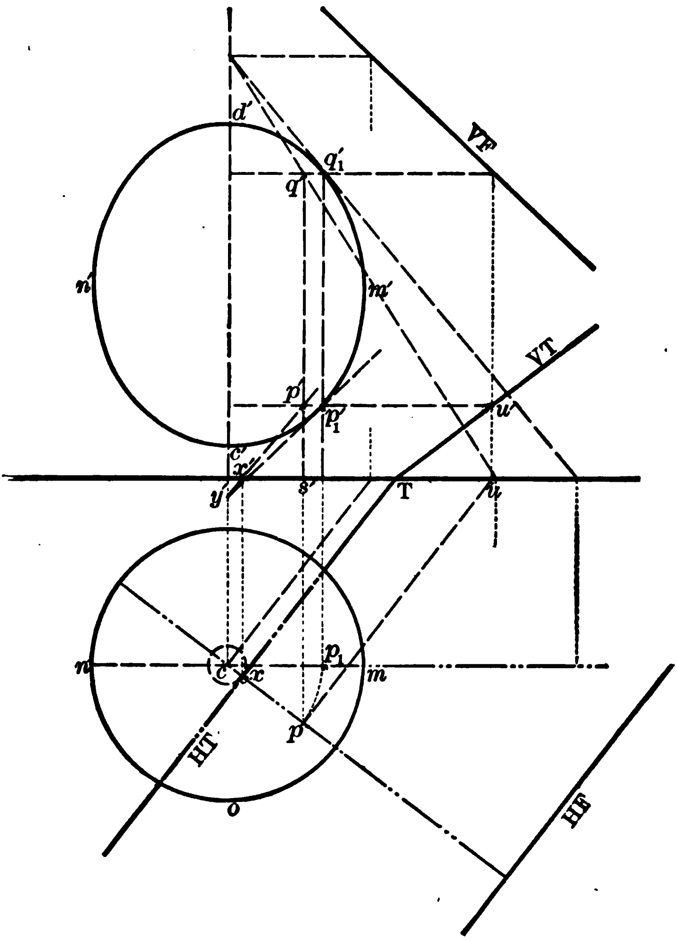


FIG. 79.

When the plane is revolved to its primitive position, the point, of which y' is the vertical projection, remains fixed, and $y'p'$ will be the vertical projection of the tangent, and cp its horizontal projection. It pierces H at x, one point of the horizontal trace of the required plane. pp_1 is the horizontal, and $p'p'_1$

the vertical projection of an arc of the circle of the surface containing P. At p draw pu perpendicular to cp. It will be the horizontal projection of the tangent to the circle at P, and $p'u'$ is its vertical projection. This pierces V at u', a point of the vertical trace. Through x draw xT parallel to pu, and through u' draw u'T; these will be the traces of the required plane.

By the same method we may pass a plane tangent to any other surface of revolution at a given point.

Since the tangent plane and horizontal plane are both perpendicular to the meridian plane through the point of contact, their intersection, which is the horizontal trace, will be perpendicular to the meridian plane and to its horizontal trace.

While at a given point on a double-curved surface only one tangent plane can be passed, it may be proved, as in Art. 162, that from a point without the surface an infinite number of such planes can be passed.

183. PROBLEM 60. **To pass a plane through a given straight line and tangent to a sphere.**

Analysis. Conceive the sphere to be circumscribed by a cylinder of revolution whose axis is parallel to the given line. The line of contact will be the circumference of a great circle whose plane is perpendicular to the axis and to the given line. A plane through the straight line tangent to this cylinder will be tangent also to the sphere. The plane of the circle of contact will intersect the given line in a point, and the required tangent plane in a straight line drawn from this point tangent to the circle. The plane of this tangent and the given line will be the required plane. Without constructing the cylinder, we have then simply to *pass a plane through the center of the sphere perpendicular to the given line, and from the point in which it intersects the line, to draw a tangent to the circle cut from the sphere by the same plane, and pass a plane through this tangent and the given line.*

Construction. The student will assume a sphere and an oblique line outside the sphere, and make the construction in accordance with the above analysis.

184. PROBLEM 61. **To pass a plane through a given straight line and tangent to any surface of revolution.**

Let the horizontal plane be taken perpendicular to the axis, of which c, Fig. 80, is the horizontal, and $c'd'$ the vertical, projection. Let poq be the intersection of the surface by the horizontal plane, $p'd'q'$ the vertical projection of the meridian curve parallel to the vertical plane, and MN the given line.

Analysis. If this line revolve about the axis of the surface, it will generate a hyperboloid of revolution of one nappe (Art. 168), having the same axis as the given surface. If we now conceive the plane to be passed through the line tangent to the surface, it will also be tangent to the hyperboloid at a point of the given straight line (Art. 141); and since the meridian plane through the point of contact on each surface must be perpendicular to the common tangent plane (Art. 175), these meridian planes must form one and the same plane. This plane will cut from the given surface a meridian curve, from the hyperboloid a hyperbola (Art. 174), and from the tangent plane a straight line tangent to these curves at the required points of contact (Art. 86). The plane of this tangent and the given line will therefore be the required plane.

Construction. Construct the hyperboloid, as in Arts. 172 and 173. yx will be the horizontal, and $x'y'w'$ the vertical, projection of one branch of the meridian curve parallel to V. If the meridian plane through the required points of contact be revolved about the common axis until it becomes parallel to V, the corresponding meridian curves will be projected, one into the curve $p'd'q'$, and the other into the hyperbola $x'y'w'$. Tangent to these curves draw $x'r'$; X will be the revolved position of the point of contact on the hyperbola (found as in Art. 69), and R

that on the meridian curve of the given surface (Art. 67). When the meridian plane is revolved to its true position, X will be horizontally projected in *mn* at *s*; *sc* will be the horizontal trace

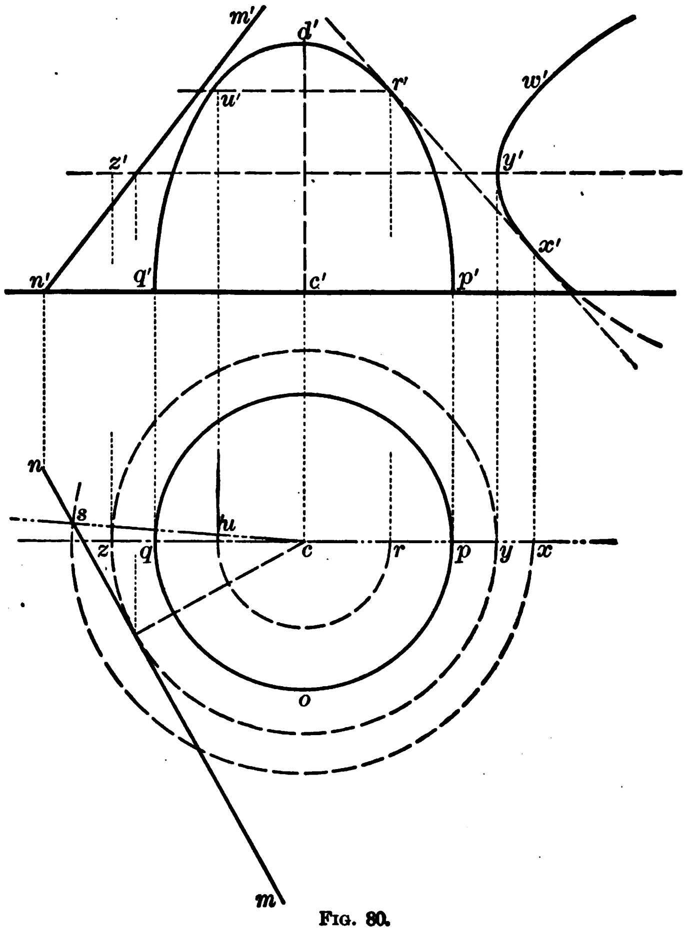


FIG. 80.

of the meridian plane, and *u* will be the horizontal, and *u'* the vertical, projection of the point of contact on the given surface. A plane through this point and MN will be the tangent plane.

185. In general, through a given straight line a limited number of planes only can be passed tangent to a double-curved surface. For let the surface be intersected by a system of planes parallel to the given line, and tangents be drawn to the sections also parallel to the line. These will form a cylinder tangent to the surface. Any plane through the straight line tangent to this cylinder will be tangent to the surface; and the number of tangent planes will be determined by the number of tangents which can be drawn from a point of the given line to a section made by a plane through this point.

Intersection of Surfaces of Revolution with Other Surfaces

186. Problem 62. **To find the intersection of any surface of revolution by a plane.**

Let the surface be a hyperboloid of revolution of one nappe given as in Fig. 81, and let T be the cutting plane.

Analysis. If a meridian plane be passed perpendicular to the cutting plane, it will intersect it in a straight line, which will divide the curve symmetrically, and be an axis. If the points in which this line pierces the surface be found, these will be the points on the axis of symmetry of the curve. Now intersect by a system of planes perpendicular to the axis of the surface; each plane will cut from the surface a circumference, and from the given plane a straight line, the intersection of which will be points of the required curve.

Construction. **To find the points on the axis of symmetry,** draw *cn* perpendicular to HT; it will be the horizontal trace of the auxiliary meridian plane. This plane intersects T in the straight line NC (Art. 40), and the surface in a meridian curve which is intersected by NC in the two vertices of the required curve.

To find these points, revolve the meridian plane about the

axis until it becomes parallel to V. The meridian curve will be vertically projected into the hyperbola which limits the vertical projection of the surface (Art. 58), and the line NC into $d'c'$. The points s' and r' will be the vertical projections

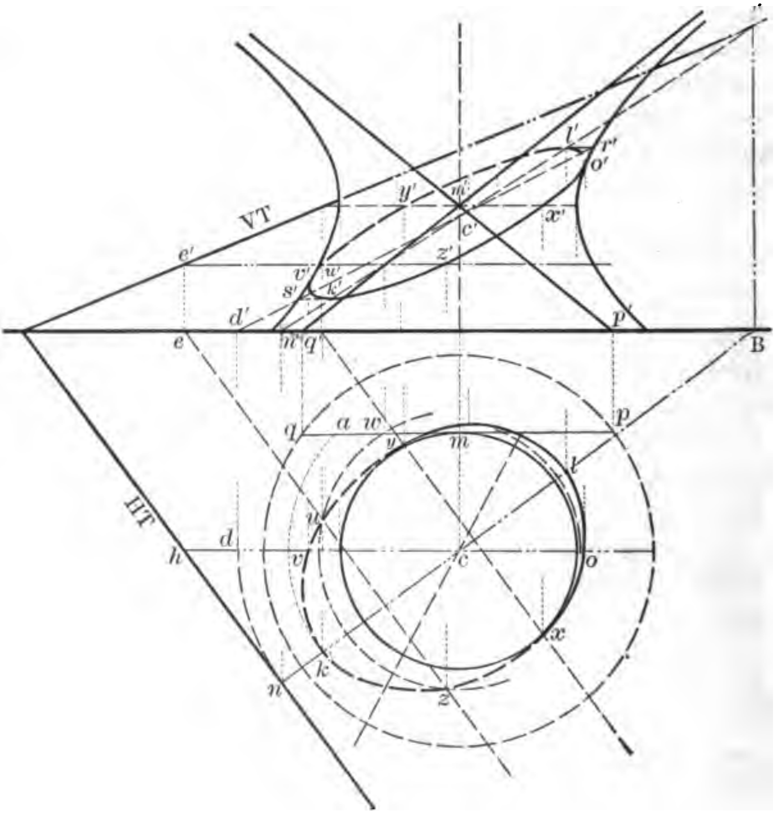


Fig. 81.

of the revolved positions of the vertices. After the counter-revolution, these points are horizontally projected at k and l, and vertically at k' and l'.

To find the points in which the vertical projection of the curve of intersection is tangent to the contour line of the surface, use

the meridian plane parallel to V. It cuts from the given plane a line horizontally projected in *ho* and vertically projected in a line (not shown) parallel to V·T. It cuts from the surface of revolution the meridian curve whose vertical projection is already drawn. These lines intersect in the points *vv'* and *oo'*, the required points of tangency.

To find other points of the curve, let *e'z'* be the vertical trace of an auxiliary plane perpendicular to the axis. It cuts the surface in a circle horizontally projected in *wuz*, and the plane in a straight line, which piercing V at *e'* is horizontally projected in *ez*; hence U and Z are points of the curve. In the same way any number of points may be determined.

The points upon any particular circle may be determined by using the plane of this circle as an auxiliary plane. If the curve crosses the circle of the gorge, the points in which it crosses are determined by using the plane of this circle, and in this case the horizontal projection of the curve must be tangent to the horizontal projection of the circle of the gorge, at the points *x* and *y*.

To draw a tangent to the curve at Z, pass a plane tangent to the surface at Z, as in Art. 182; its intersection with T will be the tangent (Art. 105). The curve may be represented in its true dimensions as in Art. 110.

Let the intersection of an oblique plane with a sphere, an ellipsoid of revolution, a torus, and a paraboloid of revolution, be constructed in accordance with the principles given above.

187. Problem 63. **To find the intersection of a cylinder and a sphere.**

Analysis. Intersect the surfaces by a system of auxiliary planes parallel to the rectilinear elements of the cylinder and perpendicular to the horizontal plane (or the plane of the base). Each plane will cut from the cylinder two rectilinear elements,

and from the sphere a circle, the intersection of which will be points of the required curve.

Construction. Revolve each auxiliary plane about its horizontal trace until it coincides with H. Find the revolved position of each circle and of the corresponding pairs of elements. They will intersect in points which, when returned to their primitive positions, will be points on the required curve of intersection.

A tangent at any point of the curve may be constructed by finding the intersection of two planes, one tangent to the cylinder and one to the sphere at that point, as in Art. 116.

Let the student assume an oblique cylinder and a sphere intersecting it, and solve the problem in accordance with the above method.

188. PROBLEM 64. **To find the intersection of two surfaces of revolution whose axes are in the same plane.**

First, let the axes intersect and let one of the surfaces be an ellipsoid of revolution and the other a paraboloid; and let the horizontal plane be taken perpendicular to the axis of the ellipsoid, and the vertical plane parallel to the axes; $(c, c'd')$, Fig. 82, being the axis of the ellipsoid, and $(cl, s'l')$ that of the paraboloid. Let the ellipsoid be represented as in Art. 179 and let $z'f'r'$ be the vertical projection of the paraboloid.

Analysis. Intersect the two surfaces by a system of auxiliary spheres having their centers at the point of intersection of the axes. Each sphere will intersect each surface in the circumference of a circle perpendicular to its axis (Art. 166), and the points of intersection of these circumferences will be points of the required curve.

Construction. With s' as a center and any radius $s'q'$, describe the circle $q'p'r'$; it will be the vertical projection of an auxiliary sphere. This sphere intersects the ellipsoid in

a circumference vertically projected in $p'q'$, and horizontally in pxq. It intersects the paraboloid in a circumference ver-

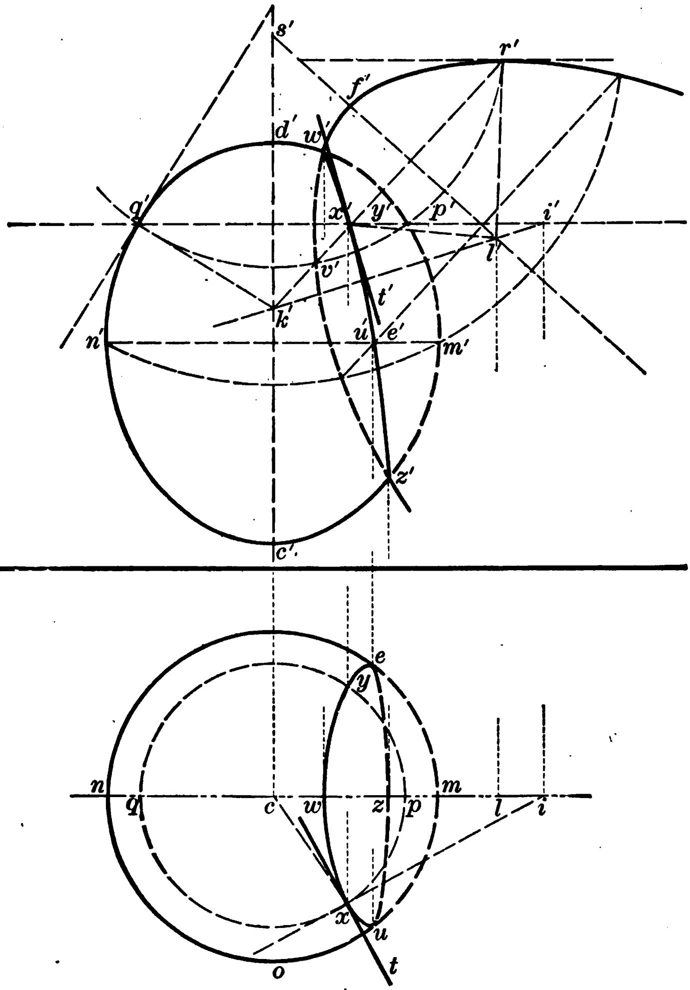


FIG. 82.

tically projected in $r'v'$. These circumferences intersect in two points vertically projected at x', y', and horizontally at x and y

In the same way any number of points may be found.

The points on the greatest circle of the ellipsoid are found by using $s'n'$ as a radius. These points are horizontally projected at u and e, points of tangency of xzy with nom.

The points W and Z are the points in which the meridian curves parallel to V intersect, and are points of the required curve.

Each point of the curve $z'x'w'$ is the vertical projection of two points of the curve of intersection, one in front and the other behind the plane of the axes.

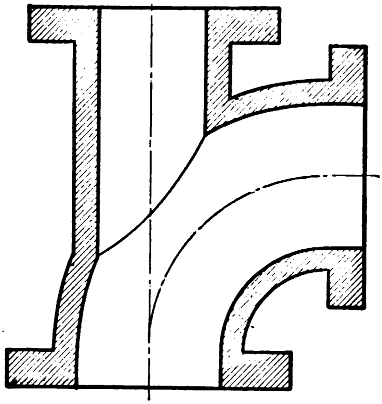


FIG. 83.—Intersection of Cylinder and Torus. Pipe Connection.

A tangent may be drawn to the curve at any point, as X, as in Art. 111.

Second. **If the axes of the two surfaces are parallel**, the construction is more simple, as the auxiliary spheres become planes perpendicular to the axes. Let the construction be made in this case.

Let it be required to find the intersection between the cylinder and the torus of Fig. 83. How would the auxiliary planes be passed?

Ex. 122. Given a torus (Art. 177), and the vertical projection of a point A on the surface.

(*a*) Find the horizontal projection of A, assuming that it lies on the side of the surface toward V.

(*b*) Pass a plane S tangent to the surface at the point A.

(*c*) How would you solve (*b*) if the surface were a sphere?

(*d*) Pass a plane T tangent to the torus and through an assumed straight line, BC.

(*e*) How would you solve (*d*) if the surface were a sphere?

(*f*) Pass a plane, U, tangent to the torus and perpendicular to the line BC.

(*g*) How would you solve (*f*) if the surface were a sphere?

(*h*) Assume the projections of a curved line cutting the surface of the torus, and find the piercing points, P and Q.

(*i*) Find the curve of intersection of the torus with a plane, an oblique cylinder or cone with circular base, a helical convolute, a warped surface with plane directer parallel to H, a helicoid, an ellipsoid of revolution whose axis intersects the axis of the torus, a paraboloid of revolution whose axis is parallel to the axis of the torus.

Problems relating to Trihedral Angles — Graphical Solution of Spherical Triangles

189. In the preceding articles we have all the elementary principles and rules relating to the orthographic projection. The student who has thoroughly mastered them will have no difficulty in their application.

Let this application now be made to the solution of the following simple problems.

190. Problem 65. **Having given two of the faces of a trihedral angle and the dihedral angle opposite one of them, to construct the trihedral angle.**

Let dsf and fse_1, Fig. 84, be the two given faces, and A the given angle opposite fse_1, dsf being in the horizontal plane and the vertical plane perpendicular to the edge sf.

Construct, as in Art. 50, de', the vertical trace of a plane whose horizontal trace is sd and which makes with H the angle A; sde' will be the true position of the required face. Revolve fse_1 about sf until the point e_1 comes into de' at e'. This must be the point in which the third edge in true position pierces V. Join $e'f$; it will be the vertical trace of the

plane of the face fse_1 in true position, and SE will be the third edge.

Revolve sde' about sd until it coincides with H; e' falls at e_2 (Art. 34), and dse_2 is the true size of the third or required face.

$e'fd$ is the dihedral angle opposite the face dse', and pvq, determined as in Art. 47, is the dihedral angle opposite dsf.

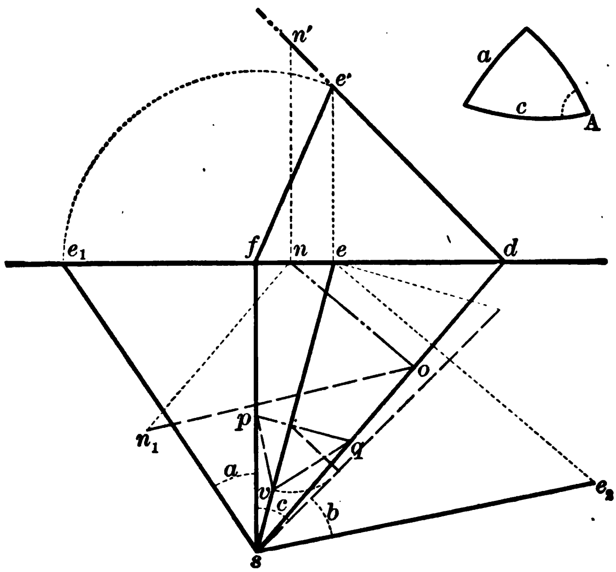


FIG. 84.

191. PROBLEM 66. **Having given two dihedral angles formed by the faces of a trihedral angle, and the face opposite one of them, to construct the angle.**

Let A and B, Fig. 85, be the two dihedral angles, and dse_2 the face opposite B.

Make $e'df$ equal to A. Revolve dse_2 about ds until e_2 comes into de' at e'. This will be the point where the edge se_2 in true position pierces V, and SE will be this edge.

Draw $e'm$, making $e'me$ equal to B, and revolve $e'm$ about $e'e$; it will generate a right cone whose rectilinear elements all make with H an angle equal to B. Through s pass the plane sfe' tangent to this cone (Art. 100). It, with the faces fsd and sde', will form the required angle.

fse_1 is the true size of the face opposite A, e_1 being the revolved position of e', determined as in Art. 34, and the third dihedral angle formed by $e'fs$ and $e'ds$ may be found as in Art. 47.

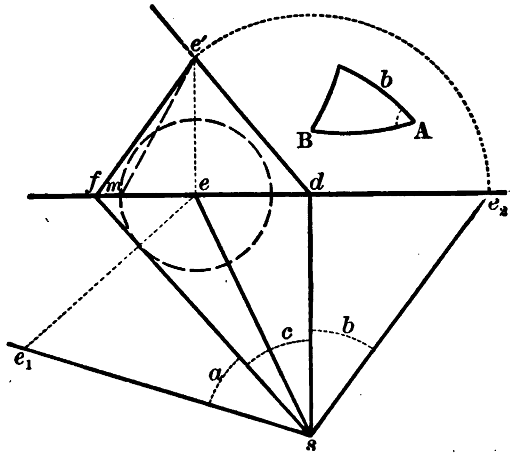


FIG. 85.

192. PROBLEM 67. **Given two faces of a trihedral angle and their included dihedral angle, to construct the angle.**

Let dse_2 and dsf, Fig. 86, be the two given faces and A the given angle.

Make $e'df$ equal to A. de' will be the vertical trace of the plane of the face dse_2 in its true position. Revolve dse_2 about sd until e_2 comes into de' at e', the point where the edge se_2 in true position pierces V. Draw $e'f$. It is the vertical trace of the plane of the third or required face, and fse_1 is its true size.

eoe_3 (Art. 49) is the dihedral angle opposite $e'ds$, and the third dihedral angle can be found as in Art. 47.

193. PROBLEM 68. **Given one face and the two adjacent dihedral angles of a trihedral angle, to construct the angle.**

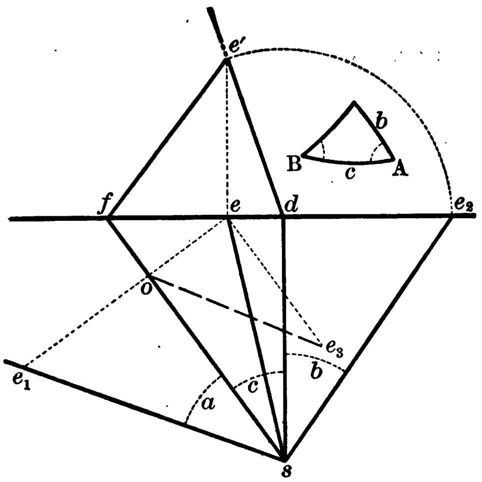


Fig. 86.

Let *dsf*, Fig. 86, be the given face, and A and B the two adjacent dihedral angles.

Make *e′df* equal to A. Construct as in Art. 50, *fe′*, the vertical trace of a plane *sfe′*, making with H an angle equal to B. SE will be the third edge, dse_2 and fse_1 the true size of the other faces, and the third dihedral angle is found as in Art. 47.

194. Problem 69. **Given the three faces of a trihedral angle, to construct the angle.**

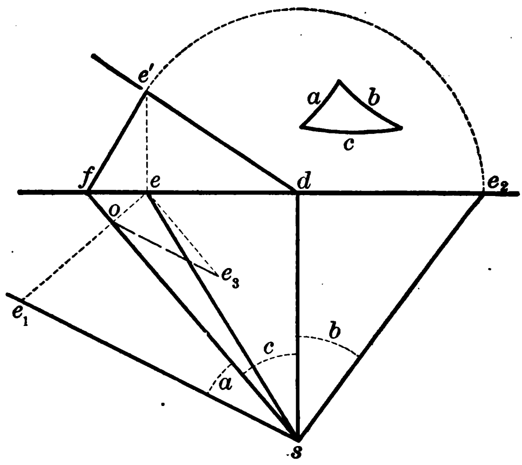


Fig. 87.

Let dse_2, *dsf*, and fse_1, Fig. 87, be the three given faces, *sd* and *sf* being the two edges in the horizontal plane.

Make se_1 equal to se_2. Revolve the face fse_1 about *fs*; e_1 describes an arc whose plane is perpendicular to H (Art. 34), of which e_1e is the horizontal and *ee′* the vertical trace. Also revolve

dse_2 about ds; e_2 describes an arc in the vertical plane. These two arcs intersect at e', the point where the third edge pierces V, and SE is this edge.

Join $e'd$ and $e'f$. These are the vertical traces of the planes of the faces dse_2 and fse_1 in true position. The dihedral angles may now be found as in the preceding articles.

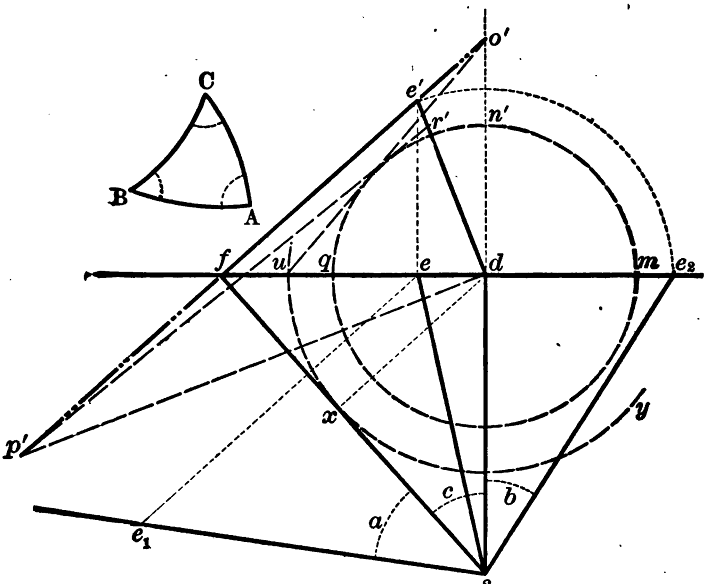


Fig. 88.

195. PROBLEM 70. **Given the three dihedral angles formed by the faces of a trihedral angle, to construct the angle.**

Let A, B, and C, Fig. 88, be the dihedral angles.

Make $e'df$ equal to A. Draw ds perpendicular to the ground line and take $e'ds$ as the plane of one of the faces. If we now construct a plane which shall make with H and $e'ds$ angles respectively equal to B and C, it, with these planes, will form the required trihedral angle.

To do this: with d as a center and any radius as dm, describe

a sphere; *mn′q* will be its vertical projection. Tangent to *mn′q* draw *o′u*, making *o′ud* equal to B. and revolve it about *o′d*. It will generate a cone whose vertex is *o′*, tangent to the sphere, and all of whose rectilinear elements make with H an angle equal to B.

Also tangent to *mn′q* draw *p′r′*, making with *de′* an angle equal to C, and revolve it about *p′d*. It will generate a cone whose vertex is *p′*, tangent to the sphere, and all of whose rectilinear elements make with the plane *sde′* an angle equal to C. If now through *o′* and *p′* a plane be passed tangent to the sphere, it will be tangent to both cones and be the plane of the required third face. *p′o′* is the vertical trace of this plane, and *fs* tangent to the base *uxy* is the horizontal trace, SE is the third edge, and *dsf*, dse_2, and fse_1 the three faces in true size.

196. By a reference to Spherical Trigonometry, it will be seen that the preceding six problems are simple constructions of the required parts of a spherical triangle when any three are given. Thus in Problem 65, *two sides a and c, and an angle A opposite one*, are given, and the others constructed. In Problem 66, *two angles A and B and a side b, opposite one*, are given, etc.

197. PROBLEM 71. **To construct a triangular pyramid, having given the base and the three lateral edges.**

Let *cde*, Fig. 89, be the base in the horizontal plane, the ground line being taken perpendicular to *cd*, and let *c*S, *d*S, and *e*S be the three edges. With *c* as a center and *c*S as a radius, describe a sphere, intersecting H in the circle *mon*. The required vertex must be in the surface of this sphere. With *d* as a center and *d*S as a radius, describe a second sphere, intersecting H in the circle *qmp*. The required vertex must also be on this surface. These two spheres intersect in a circle of which *mn* is the horizontal and *m′s′n′* the vertical projection (Art. 166). With *e* as a center and *e*S as a radius, describe a

third sphere, intersecting the second in a circle of which qp is the horizontal projection. These two circles intersect in a straight line perpendicular to H at s, and vertically projected in $r's'$. This line intersects the first circle in S, which must be a point common to the three spheres, and therefore the vertex

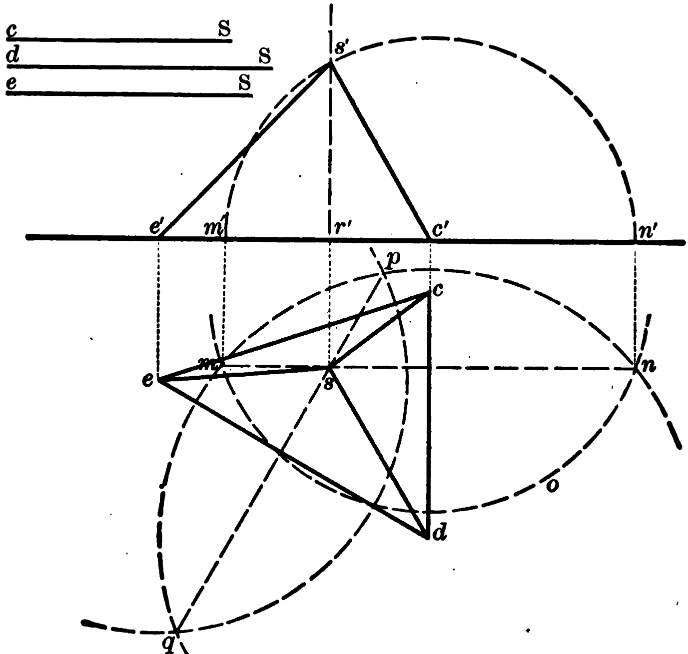


FIG. 89.

of the required pyramid. Join S with C, D, and E, and we have the lateral edges, in true position.

198. PROBLEM 72. **To circumscribe a sphere about a triangular pyramid.**

Let mno, Fig. 90, be the base of the pyramid in the horizontal plane, and S its vertex.

Analysis. Since each edge of the pyramid must be a chord of the required sphere, if any edge be bisected by a plane

perpendicular to it, this plane will contain the center of the sphere. Hence, if three such planes be constructed intersecting in a point, this must be the required center, and the radius will be the straight line joining the center with the vertex of any trihedral angle.

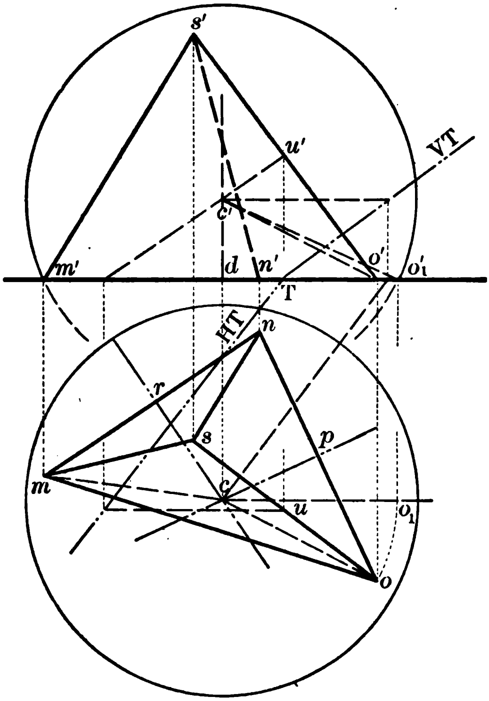


FIG. 90.

Construction. Bisect *mn* and *no*, by the perpendiculars *rc* and *pc*. These will be the horizontal traces of two bisecting and perpendicular planes. They intersect in a straight line perpendicular to H at *c*. Through U, the middle point of SO, pass a plane perpendicular to it (Art. 44). T is this plane. It is pierced by the perpendicular (c, dc') at C, which is the intersection of the three bisecting planes, and therefore the center of the required sphere. CO is its radius, the true length of which is $c'o'_1$ (Art. 26).

With c and c' as centers, and with $c'o'_1$ as a radius, describe circles. They will be, respectively, the horizontal and vertical projections of the sphere.

199. PROBLEM 73. **To inscribe a sphere in a given triangular pyramid.**

Let the pyramid S-*mno*, Fig. 91, be given as in the preceding problem.

Analysis. The center of the required sphere must be equally distant from the four faces of the pyramid, and therefore must be in a plane bisecting the dihedral angle formed by either two of its faces. Hence, if we bisect three of the dihedral angles by planes intersecting in a point, this point must be the center of the required sphere, and the radius will be the distance from the center to any face.

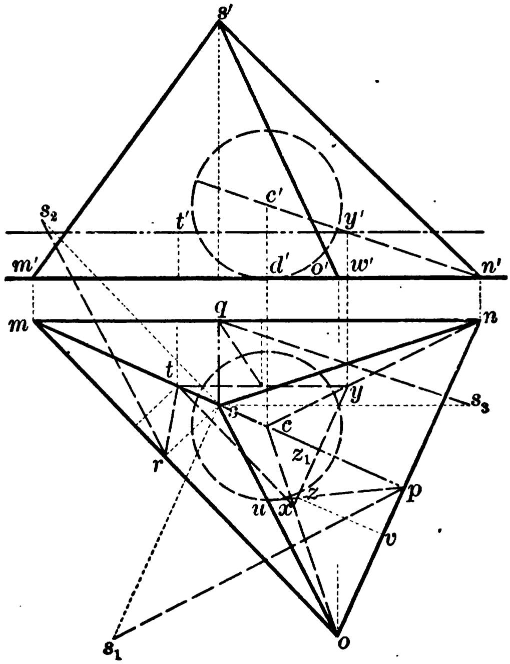


FIG. 91.

Construction. Find the angle sps_1 made by the face S*on* with H (Art. 49). Bisect this by the line *pu*, and revolve the plane sps_1 to its true position. The line *pu*, in its true position, and *on* will determine a plane bisecting the dihedral angle sps_1. In the same way determine the planes bisecting the dihedral angles srs_2, sqs_3. These planes, with the base *mno*, form a second pyramid, the vertex of which is the intersection of the three planes, and therefore the required center.

Intersect this pyramid by a plane parallel to H, whose ver-

tical trace is $t'y'$. This plane intersects the faces in lines parallel to *mn*, *no*, and *om* respectively. These lines form a triangle whose vertices are in the edges. To determine these lines, lay off *pv* equal to $y'w'$; draw *vz* parallel to *ps*. *z* will be the revolved position of the point in which the parallel plane intersects *pu* in its true position. z_1 is the horizontal projection of this point, and z_1y of the line parallel to *no*. In the same way *ty* and *tx* are determined. Draw *ox* and *ny*; they will be the horizontal projections of two of the edges. These intersect in *c*, the horizontal projection of the vertex. $n'y'$ is the vertical projection of the edge which pierces H at *n*, c' the vertical projection of the center, and $c'd'$ the radius.

With *c* and c' as centers, describe circles with $c'd'$ as a radius. They will be the horizontal and vertical projections of the required sphere.

PART II

SPHERICAL PROJECTIONS

Preliminary Definitions

200. One of the most interesting applications of the principles of Descriptive Geometry is to the *representation, upon a single plane, of the different circles of the earth's surface*, regarded as a perfect sphere.

These representations are *spherical projections.* The plane of projection, which is generally taken as that of one of the great circles of the sphere, is the *primitive plane;* and this great circle is the *primitive circle.*

The *axis of the earth* is the straight line about which the earth is known daily to revolve.

The two points in which it pierces the surface are the *poles*, one being the *north pole*, and the other the *south pole.*

The *axis of a circle* of the sphere is the straight line through its center perpendicular to its plane, and the points in which it pierces the surface are the *poles of the circle.*

The *polar distance of a point* of the sphere is its distance from either pole of the primitive circle.

The *polar distance of a circle* of the sphere is the distance of any point of its circumference from either of its poles.

201. The lines on the earth's surface usually represented are:

1. The **equator**, the circumference of a great circle whose plane is perpendicular to the axis.

2. The **ecliptic**, the circumference of a great circle making an

angle of $23\frac{1}{2}°$, nearly, with the equator. It intersects the equator in two points, called the *equinoctial points.*

3. The **meridians**, the circumferences of great circles whose planes pass through the axis, and are therefore perpendicular to the plane of the equator.

Of these meridians two are distinguished: the *equinoctial colure,* which passes through the equinoctial points; and the *solstitial colure,* whose plane is perpendicular to that of the equinoctial colure.

The solstitial colure intersects the ecliptic in two points, called the *solstitial points.*

4. **The parallels of latitude,** the circumferences of small circles parallel to the equator.

Four of these are distinguished:

The *Arctic circle,* $23\frac{1}{2}°$ from the north pole;

The *Antarctic circle,* $23\frac{1}{2}°$ from the south pole;

The *tropic of Cancer,* $23\frac{1}{2}°$ north of the equator;

The *tropic of Capricorn,* $23\frac{1}{2}°$ south of the equator.

The first two are also called *polar circles.*

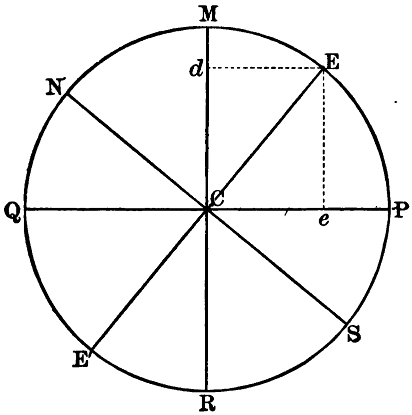


FIG. 92.

202. The *latitude* of a point on the earth's surface is its distance from the equator, measured on a meridian passing through the point.

The *horizon of a point or place* on the earth's surface is the circumference of a great circle whose plane is perpendicular to the radius passing through the point.

Let M, Fig. 92, be any point on the earth's surface. Through

this point and the axis pass a plane, and let MNS be the circumference cut from the sphere; N the north and S the south pole; ECE′ the intersection of the plane with the equator, and PCQ, perpendicular to CM, its intersection with the horizon of the given point. Then ME is the latitude of the point, and NQ the distance of the pole N from the horizon. NQ also measures the angle NCQ, the inclination of the axis to the horizon. But

$$NQ = ME,$$

since each is obtained by subtracting NM from a quadrant; that is, *the distance from either pole of the earth to the horizon of a place is equal to the latitude of that place.*

203. Let NS, Fig. 92, and MR be the axes of two circles intersecting the plane NCM in the lines EE′ and PQ respectively. ECP is then the angle made by the planes of these circles. But

$$ECP = NCM,$$

since each is obtained by subtracting MCE from a right angle; that is, *the angle between any two circles of the sphere is equal to the angle formed by their axes.*

204. If a plane be passed through the axes of any circle of the sphere and of the primitive circle, its intersection with the primitive plane is the *line of measures of the given circle.* This auxiliary plane is perpendicular to the planes of both circles, and therefore is perpendicular to their intersection; hence the line of measures, which is a line of this plane, must be *perpendicular to the intersection* of the given circle with the primitive circle, and must also *pass through the center* of the primitive circle. Thus, if EE′, Fig. 92, is the intersection of a circle with the primitive plane NESE′, NS is its line of measures. Also, NS is the line of measures of any small circle whose intersection with the primitive plane is parallel to EE′.

Orthographic Projections of the Sphere

205. When the point of sight is taken in the axis of the primitive circle, and at an infinite distance from this circle, *the projections of the sphere are orthographic* (Art. 4).

If E, Fig. 92, be any point, e will be its orthographic projection on the plane of a circle whose axis is CM. But

$$Ce = Ed;$$

that is, the orthographic projection of any point of the surface of a sphere is *at a distance from the center of the primitive circle equal to the sine of its polar distance multiplied by the radius of the sphere.*

206. The circumference of a circle, oblique to the primitive plane, *is projected into an ellipse.* For the projecting lines of its different points form the surface of a cylinder whose intersection with the primitive plane is its projection (Art. 80), and this intersection is an ellipse (Art. 110).

If the plane of the circumference be perpendicular to the primitive plane, its projection is a straight line (Art. 58).

If the plane of the circumference be parallel to the primitive plane, its projection is an equal circumference (Art. 58).

The projection of every diameter of the circle which is oblique to the primitive plane will be a straight line less than this diameter (Art. 26), while the projection of that one which is parallel to the primitive plane will be equal to itself (Prop. XIV, Art. 14). This projection will then be longer than any other straight line which can be drawn in the ellipse, and is therefore its *transverse axis.*

The projection of that diameter which is perpendicular to the one which is parallel to the primitive plane will be perpendicular to this transverse axis (Prop. XX, Art. 14), and pass through the center, and therefore be the *conjugate axis* of the ellipse (Art. 67).

This last diameter is perpendicular to the intersection of the plane of the given circle and the primitive plane, and therefore makes with the primitive plane the same angle as the circle; and one half its projection, or the *semiconjugate axis* of the ellipse, is evidently the *cosine of this inclination*, computed to the radius of the given circle. Hence to project any circle orthographically, we have simply *to find the projection of that diameter which is parallel to the primitive plane, and through its middle point draw a straight line perpendicular to it, and make it equal to the cosine of the angle made by the circle with the primitive plane.* The first line is the transverse, and the second the semiconjugate axis of the required ellipse, which may then be accurately constructed as in Art. 67. The trammel method of constructing an ellipse upon its two principal axes, described in Art. 45, may also be used to advantage in this case.

It should be remarked that the conjugate axis of the ellipse always lies on the line of measures of the circle to be projected (Art. 204).

207. The line of measures of a circle evidently contains the projections of both poles of the circle (Art. 204); and since the arc which measures the distance of either pole from the pole of the primitive circle measures also the inclination of the two circles (Art. 203), it follows that either pole of a circle is orthographically projected in its line of measures *at a distance from the center of the primitive circle equal to the sine of its inclination multiplied by the radius of the sphere* (Art. 205).

208. PROBLEM 74. **To project the sphere upon the plane of any one of its great circles.**

Let $EpE'q$, Fig. 93, be the primitive circle intersecting the equator in the points E and E′, and making with it an angle denoted by A. Let E and E′ also be assumed as the equinoctial points. The line EE′ is then the intersection of the primitive plane by the equator, ecliptic, and equinoctial colure,

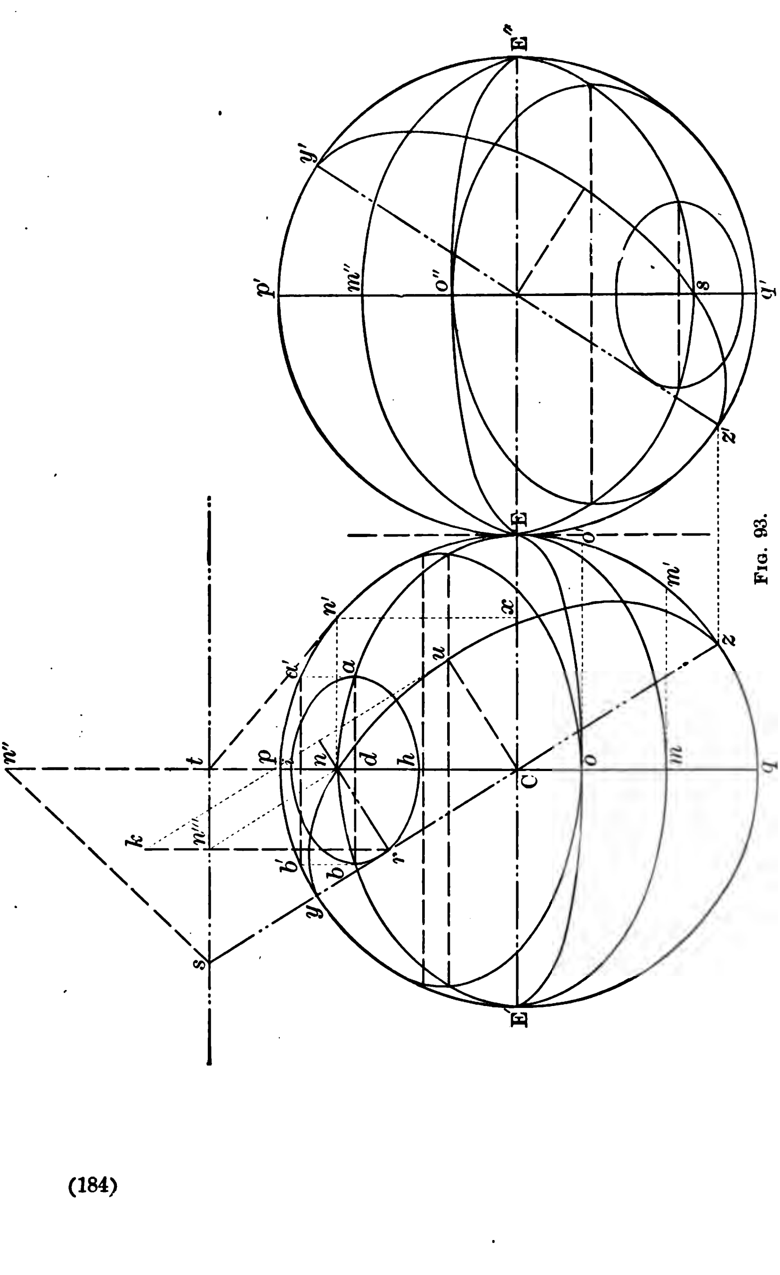


FIG. 93.

and pq perpendicular to it is the line of measures of all these circles.

Let us first project the hemisphere lying between the primitive plane and the north pole.

Since EE$'$ is that diameter of the *equator* which lies in the primitive plane, it is its own projection, and therefore the transverse axis of the ellipse into which the equator is projected. From q lay off qm' equal to A, and draw $m'm$ perpendicular to pq. $Cm = R \cos A$, and is the semiconjugate axis (Art. 206). On this and EE$'$ describe the semiellipse EmE$'$; it is the projection of that part of the equator lying above the primitive plane.

n is the projection of the *north pole*, En' being made equal to A, and Cn equal to $n'x$, which is equal to R sin A (Art. 207).

EE$'$ is also the transverse axis of the projection of the *ecliptic*. If the portion of the ecliptic on the hemisphere under consideration lies between the equator and the north pole, it will make an angle with the primitive plane greater than that of the equator by $23\frac{1}{2}°$. If it lies between the equator and the south pole, it will make a less angle by $23\frac{1}{2}°$. Taking the former case, lay off $m'o'$ equal to $23\frac{1}{2}°$, then $qo' = A + 23\frac{1}{2}°$, and $Co = R \cos (A + 23\frac{1}{2}°) =$ the semiconjugate axis, with which and EE$'$ describe the semiellipse EoE$'$, the projection of one half the ecliptic.

The *equinoctial colure*, making with the equator a right angle, makes with the primitive plane an angle equal to $90° + A$. EE$'$ is the transverse axis of its projection, and $Cn = R \cos qn' = R \cos (90° + A) =$ the semiconjugate axis. And the semiellipse EnE$'$ is the projection of that half above the primitive plane.

The *solstitial colure*, being perpendicular to the equator and equinoctial colure, is perpendicular to EE$'$, and therefore to the primitive plane; hence its projection is the straight line qp (Art. 206).

To project any *meridian*, as that which makes with the solstitial colure an angle denoted by B, pass a plane tangent to the sphere at the north pole. It will intersect the planes of the given meridian and solstitial colure in lines perpendicular to the axis and making with each other an angle equal to B; and these lines will pierce the primitive plane in points of the intersections of the planes of these meridians with the primitive plane (Prop. XXI, Art. 14). To determine this tangent plane, revolve the solstitial colure about pq as an axis until it comes into the primitive plane. It will then coincide with $En'pq$, and the north pole will fall at n'. Draw $n't$ tangent to $En'p$; it is the revolved position of the intersection of the required tangent plane by the plane of the solstitial colure. It pierces the primitive plane at t, and st perpendicular to Cn (Art. 181) is the trace of the tangent plane. Revolve this plane about ts until it coincides with the primitive plane. The north pole falls at n''; tn'' being equal to tn' (Art. 34). Through n'' draw $n''s$, making with $n''t$ an angle equal to B. This will be the revolved position of the intersection of the tangent plane by the plane of the given meridian. It pierces the primitive plane at s, and sC is the intersection of the meridian plane with the primitive, and yz is the transverse axis of the required projection (Art. 206).

To find the semiconjugate: through the north pole pass a plane perpendicular to yz; nr is its trace. Revolve this plane about nr until it coincides with the primitive plane. N falls at n''', and $n'''rn$ is the angle made by the meridian with the primitive plane (Art. 49). Lay off rk equal to CE, and draw ku parallel to yz. Cu is the required semiconjugate axis, and the semiellipse yuz is the projection of that half of the meridian which lies above the primitive plane.

Since the plane of *any parallel of latitude*, as the Arctic circle, is parallel to the equator, it will be intersected by the plane of

the equinoctial colure in a diameter parallel to EE′, and to the primitive plane, and the projection of this diameter will be the transverse axis of the projection. To determine it: revolve the equinoctial colure about EE′ as an axis until it coincides with EpE′q; N falls at p. From p lay off pa' equal to $23\frac{1}{2}°$, the polar distance of the parallel, and draw $a'b'$. It will be the revolved position of that diameter of the Arctic circle which is parallel to the primitive plane. When the colure is revolved to its true position, $a'b'$ will be projected into ab, the required transverse axis. From d, its middle point, lay off $di = \cos A$, computed to the radius da; it will be the semiconjugate axis, and the ellipse $aibh$ is the required projection.

In the same way the tropic of Cancer or any other parallel may be projected.

If the polar distance of the parallel is greater than $90° - A$, the inclination of the axis, the parallel will pass below the primitive plane and a part of its projection be drawn dashed.

The projection of the tropic of Cancer is tangent to EoE′ at o (Art. 61).

209. Each point in the primitive circle is evidently the projection of two points of the surface of the sphere, one above and the other below the primitive plane. To represent these points distinctly and prevent the confusion of the drawing, we first project the upper hemisphere, as above, and then revolve the lower 180° about a tangent to the primitive circle at E. It will then be above the primitive plane and may be projected in the same way as the first. Em''E″ is the projection of the other half of the equator; s of the south pole; Eo''E″ of the other half of the ecliptic; EsE″ of the equinoctial colure; $y'sz'$ of the meridian, etc.

210. If the projection be made on the equator, the preceding problem is much simplified. Thus, let EpE′q, Fig. 94, be the equator; n is the projection of the north pole; EoE′ of one half

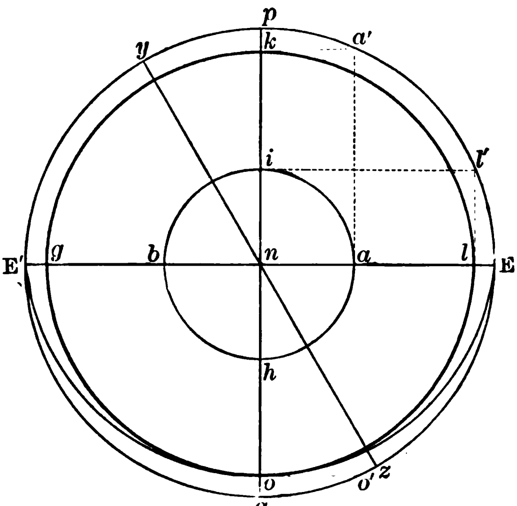


FIG. 94.

of the ecliptic, qo' being equal to $23\frac{1}{2}°$, and no its cosine.

Since the meridians are all perpendicular to the equator, EE′ is the projection of the equinoctial, and pq of the solstitial colure; yz of the meridian making an angle of 30° with the solstitial colure.

Since the parallels of latitude are parallel to the primitive plane, $ahbi$ is the projection of the Arctic circle, and $ogkl$ that of the tropic of Cancer, na being equal to the sine of $23\frac{1}{2}°$, and nl equal to sin $66\frac{1}{2}°$ (Art. 205). $ogkl$ is tangent to EoE′ at o.

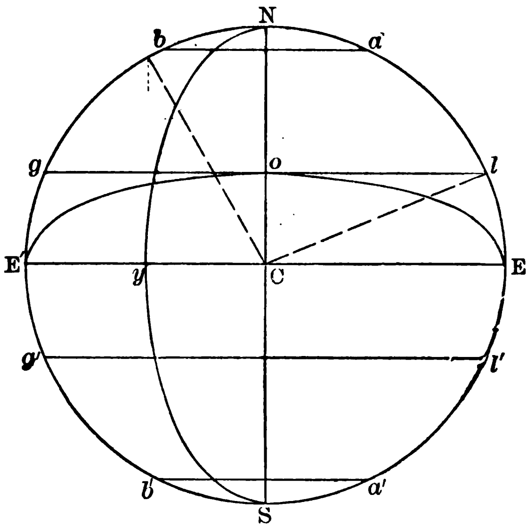


FIG. 95.

211. If the projection be made on the equinoctial colure, let ENE′S, Fig. 95, be the primitive circle, and E and E′ the equinoctial points.

Since the equator is perpendicular to the primitive plane, EE′ will be its projection. N is the north and S the south pole. E*o*E′ is the projection of one half the ecliptic; C*o* being equal to cos $66\frac{1}{2}°$. NS is the projection of the solstitial colure.

Since the parallels are perpendicular to the primitive plane, *ab* and *a′b′* are the projections of the polar circles, N*a* and S*a′* being each equal to $23\frac{1}{2}°$; and *lg* and *l′g′* the projections of the tropics, N*g* and S*g′* being each equal to $66\frac{1}{2}°$.

N*y*S is the projection of one half the meridian making an angle of 30° with the solstitial colure, or 60° with the primitive plane, C*y* being equal to cos 60°, or $\frac{1}{2}$ CE′.

212. The projections may be made upon the ecliptic and horizon of a place, in the same way as in Problem 73. In the former case, the angle A will equal $23\frac{1}{2}°$; and in the latter, since the angle included between the axis and the horizon is equal to the latitude of the place (Art. 202), the angle A between the equator and the horizon will be 90° + the latitude.

Stereographic Projections of the Sphere

213. The natural appearance and beauty of a scenographic drawing will depend very much upon the position chosen by the draftsman or artist for the point of sight. This should be so selected that a person taking the drawing into his hand for examination will naturally place his eye at this point. From any other position of the eye the drawing will appear to some extent distorted. Hence it is that an orthographic drawing never appears perfectly natural, as it is impossible to place the eye of the observer at an infinite distance from it.

In spherical projections, if the point of sight be taken at either pole of the primitive circle, the projections are *stereographic*, and, in general, present the best appearance to the eye of an ordinary observer, as in this case the projections of all circles of the sphere, as will be seen in Art. 216, are circles.

214. The projection of each point on the surface of the sphere will be that point in which a straight line, through it and the point of sight, pierces the primitive plane (Art. 4).

Let M, Fig. 96, be any point on the surface of the sphere. Through it and the axis of the primitive circle pass a plane. It will intersect the sphere in a great circle EME′S, and the primitive plane in a straight line EE′; N and S being the poles of the primitive circle, and S the point of sight. NM is the polar distance, and *m* the stereographic projection of M. C*m* is the tangent of the arc C*o*, computed to the radius CS = CE, and C*o* is one half of NM (plane geometry). That is, the stereographic projection of any point of the surface of the sphere is at a *distance from the center of the primitive circle equal to the radius of the sphere multiplied by the tangent of one half its polar distance.*

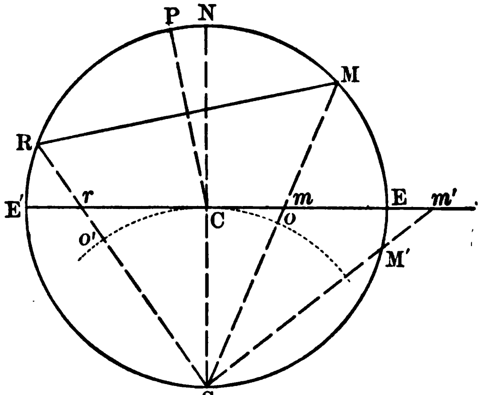


FIG. 96.

In this projection it should be observed that the polar distance of a point is always its distance from the pole opposite the point of sight, and often exceeds 90°.

215. If a plane be passed through the vertex of an oblique cone with a circular base, perpendicular to this base and through its center, such a plane is a *principal plane*, and evidently bisects all chords of the cone drawn perpendicular to it.

Let SAB, Fig. 97, be such a plane intersecting the cone in

the elements SA and SB, and the base in the diameter AB. If this cone be now intersected by a plane, T, perpendicular to the principal plane, and making with one of the principal elements, as SA, an angle, S*ba*, equal to the angle SBA, which the other makes with the plane of the base, the section is a *subcontrary section* and will be the circumference of a circle. For, through *o*, any point of *ba*, which is the orthographic projection of the curve of intersection on the principal plane, pass a plane parallel to the base. It cuts from the cone the circumference of a circle, and intersects the plane of the subcontrary section in a straight line perpendicular to SAB at *o*, and the two curves have at this point a common ordinate. The similar triangles *aod* and *cob* give the proportion

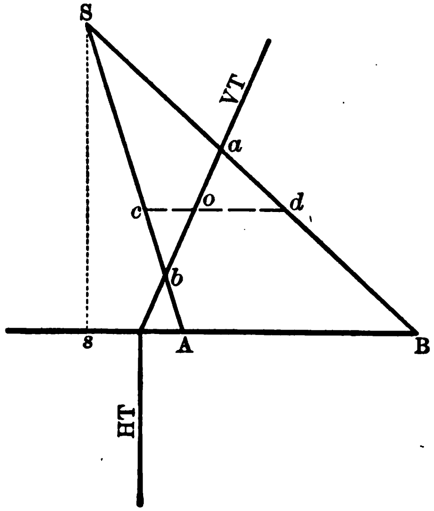


FIG. 97.

$$ao : oc :: od : ob; \text{ or, } ao \times ob = oc \times od.$$

But $oc \times od$ is equal to the square of the common ordinate, since the parallel curve is a circle; hence $ao \times ob$ is equal to the square of the ordinate of the subcontrary section, which must, therefore, be *a circle*.

216. To project any circle of the sphere, through its axis and the axis of the primitive circle pass a plane, and let ENE′S, Fig. 96, be the circle cut from the sphere by this plane; S the point of sight; RM the orthographic projection of the given circle on the cutting plane (Art. 58); CN the axis of the

primitive circle orthographically projected in EE′; and CP the axis of the given circle.

The projecting lines, drawn from points of the circumference to S, form a cone whose intersection by the primitive plane will evidently be the stereographic projection of the circumference; SRM is the principal plane of this cone, and SR and SM are the principal elements. The primitive plane is perpendicular to this plane, and intersects the cone in a curve of which *rm* is the orthographic projection. But the angle

$$\mathrm{S}rm = \mathrm{SMR}.$$

Since each is measured by SR, ES being equal to E′S, hence this section is a subcontrary section, and therefore a circle whose diameter is *mr*. That is, *the stereographic projection of every circle on the surface of a sphere* whose plane does not pass through the point of sight *is a circle.*

mr is also the line of measures of the given circle (Art. 204) and evidently contains the *center of its projection.*

The distance

$$\mathrm{C}r = \tan \mathrm{C}o' = \tan \tfrac{1}{2}(\mathrm{PR} + \mathrm{PN}),$$

and

$$\mathrm{C}m = \tan \mathrm{C}o = \tan \tfrac{1}{2}(\mathrm{PR} - \mathrm{PN}).$$

Hence the extremities of *a diameter of the projection* of any circle on the surface of the sphere are in its line of measures, one *at a distance from the center of the primitive circle, equal to the tangent of one half the sum of the polar distance and inclination* of the circle, and the other *at a distance equal to the tangent of one half the difference of these two arcs.*

When the polar distance is greater than the inclination, these extremities will evidently be on different sides of the center of the primitive circle. When less, they will be on the same side. If the polar distance is equal to the inclination, the projection of the given circle will pass through the center of the primitive circle.

The polar distance and inclination of any circle being known, a diameter of its projection can thus be constructed, and thence the projection.

217. If the circle be parallel to the primitive plane, the subcontrary and parallel sections coincide, and the projection is a circle whose center is at the center of the primitive circle, and radius the distance of the projection of any point of the circumference from the center of the primitive circle; that is, *the tangent of half the circle's polar distance* (Art. 214).

If the plane of the circle pass through the point of sight, the projecting cone becomes a plane, and *the projection is a straight line.*

218. If a straight line be tangent to a circle of the sphere, its projection will be tangent to the projection of the circle. For the projecting lines of the circumference form a cone, and those of the tangent a plane tangent to this cone along the projecting line of the point of contact; hence the intersections of the cone and plane by the primitive plane are tangent to each other at the projection of the point of contact (Art. 91). But the first is the projection of the circle, and the second that of the tangent.

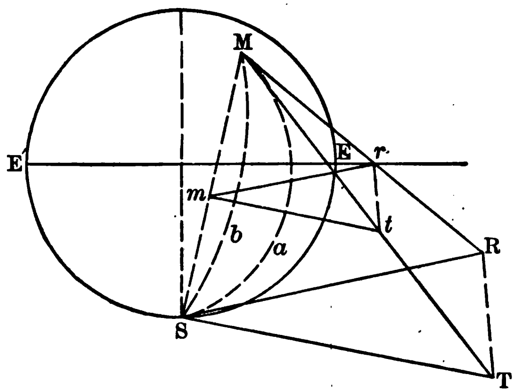


FIG. 98.

219. Let MR and MT, Fig. 98, be the tangents to two circles of the sphere at a common point M. Let these tangents be projected on the primitive plane by the planes RMS and TMS respectively, in the lines *mr* and *mt*, and let M*a*S and M*b*S be

the circles cut from the sphere by these planes, and let SR and ST be the lines cut from the plane tangent to the sphere at S. Since this tangent plane is parallel to the primitive plane, the lines SR and ST will be parallel respectively to *mr* and *mt*, and the angle RST $=$ *rmt*. Join RT. Since RS and RM are each tangent to the circle M*a*S, they are equal, and for the same reason TM $=$ TS; hence the two triangles RMT and RST are equal, and the angle

$$\text{RMT} = \text{RST} = rmt\,;$$

that is, *the angle between any two tangents to circles of the sphere at a common point is equal to the angle of their projections.*

The angle between the circles is the same as that between their tangents; and since the projections of the tangents are tangent to the projections of the circles, the angle between the projections of the circles is the same as that between the projections of the tangents; hence *the angle between any two circumferences or arcs is equal to the angle between their projections.*

220. If from the centers of the projections of two circles radii be drawn to the intersection of these projections, they will make the same angle as the circles in space. For these radii, being perpendicular to the tangents to the projections at their common point, make the same angle as these tangents, and therefore as the projections of the arcs, or as the arcs themselves.

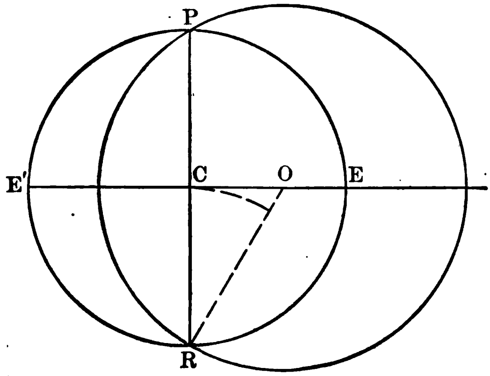


Fig. 99.

221. If the circle to be projected be a *great circle*, it will intersect the primitive circle in a diameter perpendicular to its

line of measures (Art. 204). Let O, Fig. 99, be the center *of the projection* of such a circle intersecting the primitive circle EPE′R in the diameter PR, CE being its line of measures, and P and R evidently points of the projection. Draw the radius OR. The primitive circle is its own projection; therefore the angle CRO is equal to the angle between the given and primitive circles (Art. 220). CO is the tangent of this angle, and OR its secant. Hence the *center of the projection of a great circle* is in its line of measures (Art. 216), at a distance from the center of the primitive circle equal to the *tangent of its inclination*, and the radius of the projection is the *secant of this angle.*

222. Let O, Fig. 100, be the center *of the projection of a small circle* perpendicular to the primitive plane and intersecting it in PR. OC is its line of measures, and P and R points of the projection. Join CR and OR. OR is perpendicular to CR, since EPE′R is the projection of the primitive circle, and

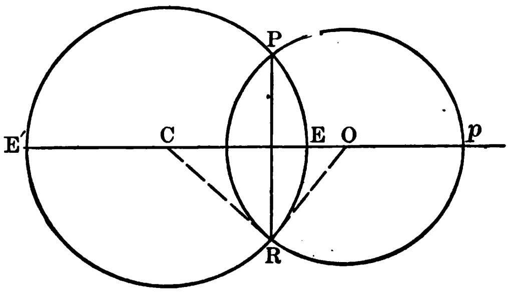


FIG. 100.

PpR that of the given circle, at right angles with it (Art. 220). OR is therefore the tangent of the arc ER, the polar distance of the given circle, and CO is its secant. Hence *the center of the projection of a small circle, perpendicular to the primitive plane,* is in its line of measures, at a distance from the center of the primitive circle equal to the *secant of the polar distance,* and the radius of the projection is the *tangent of the polar distance.*

223. Let P and R, Fig. 101, be the poles of any circle of the sphere; EPE′S being the circle cut from the sphere by the plane of the axes of the given and primitive circles, and MQ, the intersection of this plane with that of the given circle, and EE′, its intersection with the primitive circle, the line of measures of the given circle. *p* is the projection of P, and *r* of R. C*p* is the tangent of C*o*, equal to one half of NP, which measures the inclination of the given circle to the primitive (Art. 198). C*r* is the tangent of C*o*′, the complement of C*o*, or is the cotangent of C*o*.

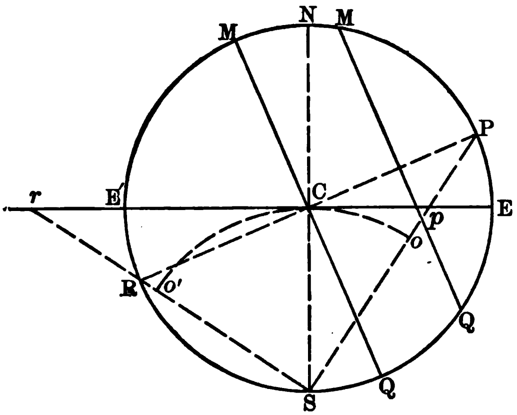


FIG. 101.

Hence the *poles of any circle* of the sphere are projected into the line of measures, the one furthest from the point of sight at a distance from the center of the primitive circle equal to the *tangent of half the inclination*, and the other at a distance equal to the *cotangent of half the inclination* of the given to the primitive circle.

224. PROBLEM 75. **To project the sphere upon the plane of any one of its great circles, as the ecliptic.**

Let E*p*E′*q*, Fig. 102, be the primitive circle intersecting the equator in EE′. In this case, E and E′ will be the equinoctial points, and in any other case may be taken as such. EE′ will also be the intersection of the plane of the equinoctial colure with the primitive plane, and *pq* the line of measures of both these circles.

We will first project the hemisphere above the primitive plane, the point of sight being at the pole underneath.

Since the *equator* makes an angle of $23\frac{1}{2}°$ with the primitive plane, we draw $E'o$, making the angle $CE'o$ equal to $23\frac{1}{2}°$. Co is the tangent of this angle and $E'o$ the secant; hence with o as a center and $E'o$ as a radius, describe the arc EmE'; it is the projection of the part of the equator above the primitive plane (Art. 221).

From E lay off Ek equal to $23\frac{1}{2}°$ and draw kE'; Cn is the tangent of half the inclination of the equator, and n is the projection of the *north pole* (Art. 223).

The equinoctial colure makes with the primitive plane an angle equal to $90°+23\frac{1}{2}°$. Through E' draw $E'x$ perpendicular to $E'o$. The angle $CE'x=90°+23\frac{1}{2}°$, and Cr, equal to tan $CE'r$, is its tangent, and $E'r$ its secant. With r as a center and $E'r$ as a radius describe $E'nE$. It is the projection of the half of the equinoctial colure above the primitive plane. It must pass through n.

The solstitial colure, being perpendicular to EE', passes through the point of sight and is projected into the straight line pq.

To project any other meridian, as that which makes an angle of $30°$ with the equinoctial colure, produce the arc $E'nE$ until it intersects Cr produced. The point of intersection, which we denote by s, will be the projection of the south pole, and since all the meridians pass through the poles, their projections will pass through n and s, and ns will be a chord common to the projections of all the meridians. If at its middle point r, the perpendicular rl be erected, this will contain the centers of all these projections. If through n, nl be drawn making rnl equal to $30°$, it will be the radius of the projection of that meridian which makes an angle of $30°$ with the equinoctial colure, since rn is the radius of the projection of this colure (Art. 220); and

l is the center of the projection of the required meridian, and ynz the projection.

To project a parallel of latitude, as the Arctic circle, lay off Ei equal to 47°, the sum of the inclination, $23\frac{1}{2}°$, and the polar distance $23\frac{1}{2}°$ (Art. 216). Draw $E'i$; $Co = \tan \frac{1}{2} Ei$, and o is one extremity of a diameter of the projection. Since the inclination is equal to the polar distance, the other extremity is at C (Art. 216), and the circle on Co, as a diameter, is the required projection.

For the tropic of Cancer, lay off Ep equal to $23\frac{1}{2}° + 66\frac{1}{2}°$; Cp is the tangent of half Ep, and p one extremity of a diameter of the projection. From k lay off kh equal to $66\frac{1}{2}°$, the polar distance of the parallel. Then $Eh = 43°$ equals the difference between the polar distance and the inclination, and Cv is the tangent of its half, and v the other extremity of the diameter, and the circle on pv the required projection. The projection of this tropic is tangent to the ecliptic at p (Art. 61).

225. Since each point on the hemisphere below the primitive plane has a greater polar distance than 90° and will therefore be projected without the primitive circle (Art. 214), and those circles near the point of sight will thus be projected into very large circles, we make a more natural representation of this hemisphere by revolving it 180°, as in the orthographic projection, about a tangent at E, the point of sight being moved to the pole of the primitive circle in its new position. The hemisphere is then above the primitive plane, and is projected as in the preceding article, s being the projection of the south pole, $Em''E''$ of the other half of the equator, EsE'' of the equinoctial colure, etc.

226. If the projection be made on any other great circle than the *ecliptic*, as on that making with the equator an angle denoted by A, the construction will be the same, the angle A being used instead of $23\frac{1}{2}°$.

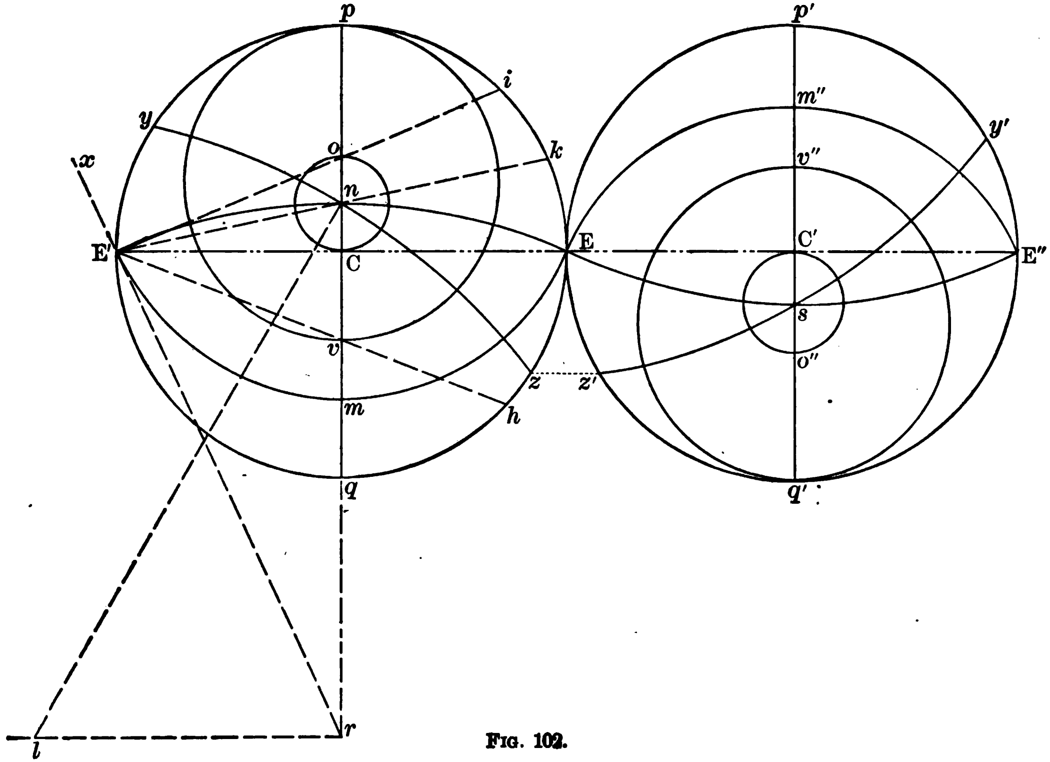


FIG. 102.

If on the horizon of a place, A must equal 90° minus the latitude (Art. 202).

227. If the projection be made on the *equator*, the preceding problem is much simplified. Thus let E*p*E′*q*, Fig. 103, be the equator, E and E′ the equinoctial points. EE′ is the intersection of the plane of the ecliptic with that of the equator, and *pq* is its line of measures and E*o*E′ its projection, *m* being the center and *mn* being equal to tan $23\frac{1}{2}°$.

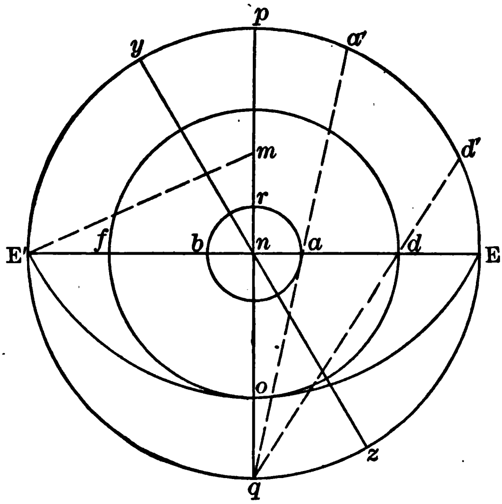


FIG. 103.

Since the *meridians* pass through the point of sight, they are projected into straight lines. EE′ is the projection of the equinoctial and *pq* of the solstitial colure; and *ynz* of the meridian which makes an angle of 30° with the solstitial colure, *n* being the projection of the north pole.

The parallels of latitude, being parallel to the equator, are projected as in Art. 217, the *Arctic circle* into *arb*, and the *tropic of Cancer* into *dof*, *na* being equal to tan $\frac{1}{2}$ $(23\frac{1}{2})°$, and *nd* equal to tan $\frac{1}{2}$ $(66\frac{1}{2})°$.

228. If the projection be on the solstitial colure, let ENE′S, Fig. 104, be the primitive circle, and EE′ its intersection with the plane of the equator.

The equator, being perpendicular to the primitive plane, passes through the point of sight and is projected into EE′.

The ecliptic, for the same reason, is projected into oo', oCE being equal to $23\frac{1}{2}°$.

NyS is the projection of the *meridian*, making with the primitive plane an angle of 30°, m being its center and Cm being equal to tan 30°.

adb and $a'hb'$ are the projections of the *polar circles*, Cx and Cx' being each equal to the secant of $23\frac{1}{2}°$, and xb and $x'b'$ each equal to the tangent of $23\frac{1}{2}°$ (Art. 222).

ogf and $o'g'd'$ are the projections of the *tropics*, each described with a radius equal to the tan $66\frac{1}{2}°$.

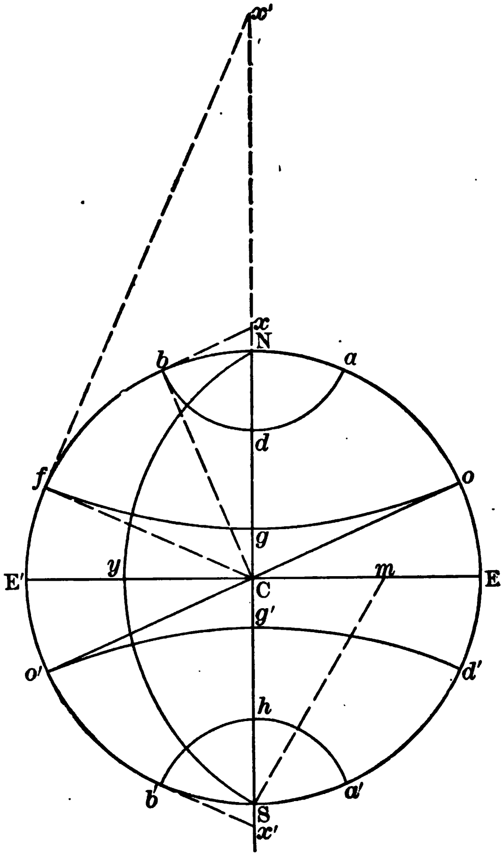


FIG. 104.

Globular Projections

229. By an examination of an orthographic or stereographic projection, it will be observed that the projections of equal arcs of great circles which pass through the pole of the primitive circle are very unequal in length. In the orthographic, as the arc is removed from the pole, its projection is diminished, and when near the primitive circle

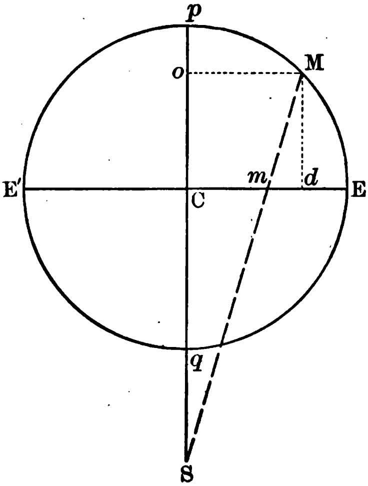


Fig. 105.

becomes very small, while the reverse is the case in the stereographic.

To avoid this inequality as far as possible, the point of sight is taken *in the axis of the primitive circle, without the surface, and at a distance from it equal to the sine of* 45° multiplied by the radius of the sphere, or $R\sqrt{\frac{1}{2}}$.

Spherical projections with this position of the point of sight are called *globular*.

Thus let the quadrant Ep, Fig. 105, be bisected at M, and let S be the point of sight. M is projected at m, and Cm will be equal to mE. For we have the proportion:

$$oS : oM :: CS : Cm; \text{ whence } Cm = \frac{CS \times oM}{oS}. \qquad (1)$$

Since $$oM = oC = qS = R\sqrt{\tfrac{1}{2}},$$

we have $$oS = R + 2\,R\sqrt{\tfrac{1}{2}}, \text{ and } CS = R + R\sqrt{\tfrac{1}{2}}.$$

Substituting these in (1) and reducing, we have

$$Cm = \frac{R}{2} = mE;$$

and it will be found that the projections of any other two equal arcs of this quadrant are very nearly equal. This is the only advantage of this mode of projection, as the projections of the circles of the sphere, being the intersections of their projecting cones with the primitive plane, will, in general, be ellipses.

Gnomonic Projection

230. If the sphere be projected on a tangent plane at any point, *the point of sight being at its center*, the projection is called *gnomonic.*

In this case the projections of all meridians are straight lines, since their planes pass through the point of sight.

If the point of contact be on the equator, the projections of the parallels of latitude will be arcs of hyperbolas (Art. 113).

If the point of contact be at either pole of the earth, these projections will be circles.

By this mode of projection the portions of the sphere distant from the point of contact will be very much exaggerated.

Cylindrical Projection

231. If a cylinder be passed tangent to a sphere along the equator, and the point of sight be taken at the center of the sphere, and the circles of the sphere be projected on the cylinder, and the cylinder be then developed, we have a developed projection called the *cylindrical projection.*

In this case the meridians will be projected into straight lines, elements of the cylinder, which are developed into parallel lines perpendicular to the developed equator (Art. 109); and the parallels into circles which are developed into straight lines perpendicular to the developed meridians and at distances from the equator each equal to the tangent of the latitude of the parallel.

Conic Projection

232. If a cone be passed tangent to a sphere along one of its parallels of latitude, and the circles of the sphere be projected on it, the point of sight being at the center, and the cone be then developed, we have a developed projection called the *conic projection.*

In this case the meridians will be projected into straight lines, elements of the cone, which are developed into straight lines passing through the vertex; and the parallels into circles whose developments will be arcs described from the vertex, each with a radius equal to the distance of the projection from the vertex (Art. 114).

Thus, in Fig. 106, let EPE′ be a section of the sphere and V the vertex of the cone tangent along the parallel of which *ab* is the orthographic projection. The equator will be projected into a circle whose diameter is *ee′*, and the parallel MN into one whose diameter is *mn*, the first being developed into an arc of which V*e* is the radius, and the second into one of which V*m* is the radius. The radius V*a* of the development of the circle of contact is evidently the tangent of its polar distance.

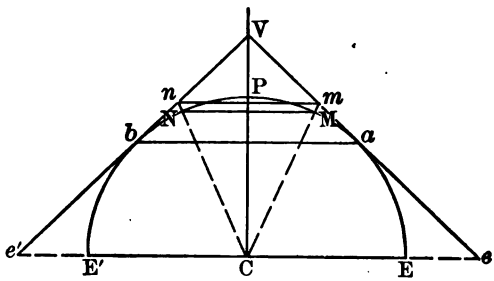


FIG. 106.

The drawing in this, as in the cylindrical projection, for those portions of the sphere distant from the circle of contact will evidently greatly exaggerate the parts projected.

233. This exaggeration may be lessened by making the projection on a cone passing through two circumferences equally distant, one from the equator and the other from the pole. Thus let *ab* and *cd*, Fig. 107, represent two circles

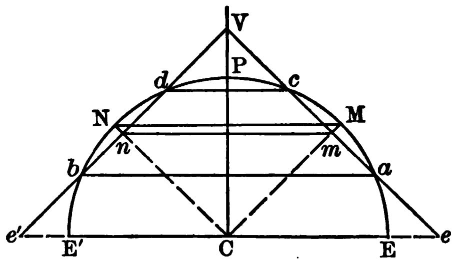


FIG. 107.

equally distant from EE′ and P, E*a* being one fourth of the quadrant; in this case, while the parts E*a* and *c*P will be exaggerated in projection, the part *ac* will be lessened.

234. When a small portion of the surface between two given parallels is to be represented, the conic method may be well used, taking the cone tangent to the sphere along a parallel midway between the given parallels if the first method be used, or passing through two parallels each distant from a limiting parallel one fourth the arc of the given portion, if the second method be used.

Construction of Maps

235. If it be desired to represent the entire surface of the earth by a map, any of the preceding methods may be used. In this case it is usual to divide the quadrant from the pole to the equator into *nine* equal parts, and to project the parallels of latitude through each of these points, as well as the polar circles and tropics; and also to divide the semiequator into *twelve* equal parts and to project the meridians passing through these points. These meridians will be 15° apart and are called *hour circles*.

The projections of the different points to be represented are then made and the map filled up in detail.

236. The stereographic projection gives the most natural representation, and in general is of the easiest construction.

In the globular projection, when the equator is taken as the primitive circle, the projections of the meridians are straight lines, and of the parallels of latitude, circles; and this projection has the advantage that these parallels, which are equally distant in space, have their projections also very nearly equally distant.

237. A very simple construction when the primitive circle is a meridian is sometimes made thus: Divide the arcs EN, E′N,

ES, and E′S, Fig. 108, each into *nine* equal parts, and the radii CN and CS also each into nine equal parts; then describe arcs of circles through each of the three corresponding points of division for the representations of the parallels.

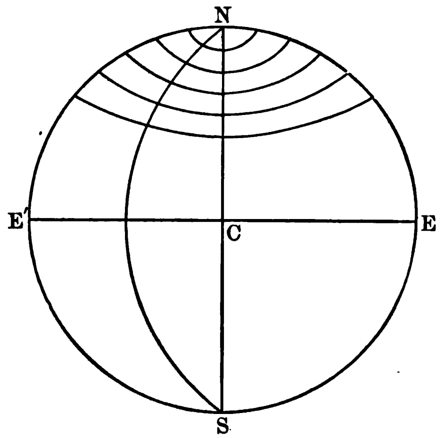


FIG. 108.

In the same way divide CE and CE′ each into *six* equal parts, and through the points of division and the poles N and S describe arcs of circles for the representations of the meridians.

This representation, though called the *equidistant projection*, is not strictly a projection. It differs little, however, from the globular projection.

LORGNA'S MAP

238. A map of the globe is sometimes made by describing a primitive circle with a radius equal to $R\sqrt{2}$, R being the radius of the sphere, and regarding this as the representation of the equator. Through its center draw straight lines making angles with each other of 15°, for the meridians.

To represent the parallels: With the center of the primitive circle as a center, and with a radius equal to $\sqrt{2Rh}$, h being the altitude of the zone, included between the pole and the parallel, describe a circle to represent each parallel in succession.

The area of the primitive circle is evidently equal to the area of the hemisphere, and the area of each other circle to that of the corresponding zone. Hence, the area between any two

circumferences will be equal to that of the zone included between the corresponding parallels.

Also the area of any quadrilateral formed by the arcs of two meridians and two parallels will be equal to its representation on the primitive plane.

MERCATOR'S CHART

239. Mercator's chart, which is much used by navigators, is a modification of the cylindrical projection. It has the great advantage that the course of a ship on the surface of the sphere, which makes a constant angle with the meridians intersected by it, will be represented on the chart by a straight line.

To secure this, as the length of the representation of a degree of longitude, as compared with the arc itself, is manifestly increased as the distance from the equator increases, it is necessary that the representations of the degrees of latitude should increase in the same ratio.

But the length of a degree of longitude, at any latitude, is known to be equal to the length of a degree at the equator multiplied by the cosine of the latitude, and since the representation of this degree at all latitudes is constant on the chart, being the distance between two parallel lines, the representations of two consecutive meridians, it follows that as we depart from the equator, this representation, as compared with the arc itself, increases as the cosine of the latitude decreases, or increases as the secant increases, and hence the representation of the degree of latitude must increase in the same ratio; that is, this representation, at any given latitude, must *equal its length at the equator multiplied by the natural secant of this latitude.*

By adding the representations of the several degrees, or, still more accurately, of the several minutes of a quadrant, the distance of the representation of each parallel from the

equator may be found, and the chart may then be thus constructed. Draw a straight line to represent the equator, then a system of equidistant parallel lines for the meridians; on any one of these lay off the proper distances computed as above for each parallel to be represented, and through the extremities of these distances draw straight lines perpendicular to the system first drawn. They represent the parallels.

240. All the maps constructed as in the preceding articles, though giving a general representation of the relative position of objects on the earth's surface, are defective in this, that there is no definite relation between actual distances of points and the representation of these distances on the maps, so that there can be no scale on the map by which these actual distances can be determined.

As in detailed representations of smaller portions of the surface this scale is absolutely necessary, other modes of constructing these maps have been devised, by which a near approximation to an accurate scale is made.

Flamstead's Method

241. In Flamstead's method, modified and improved, and now in very general use, a straight line, AB, Fig. 109, is drawn, which represents the rectified arc of the meridian passing through the middle of the portion to be mapped. A point, *c*, is then assumed to represent the point in which the parallel midway between the extreme parallels intersects this meridian, and from this point, in both directions, equal distances, *cx*, *cy*, etc., are laid off, each representing the true length of one degree of the meridian. Then with a radius equal to the tangent of the polar distance of this central parallel, the arc *dce* is described to represent the parallel. This arc is the development of the line of contact of a cone tangent to the sphere. Also, with the same center, arcs are described

through each of the points of division x, y - - - r, o, to represent the other parallels one degree distant from each other. Then, on each of these arcs, from AB, lay off both ways the arcs ca, ca', yv, yv', os, os', etc., each equal to the length of a degree of longitude at the points c, o, etc., viz. the length of a degree at the equator multiplied by the natural cosine of the latitude of the point. Through the points a, v, s, etc., and a', v', s', etc., draw the lines avs and $a'v's'$. They will represent the two meridians making an angle of one degree each with the central meridian.

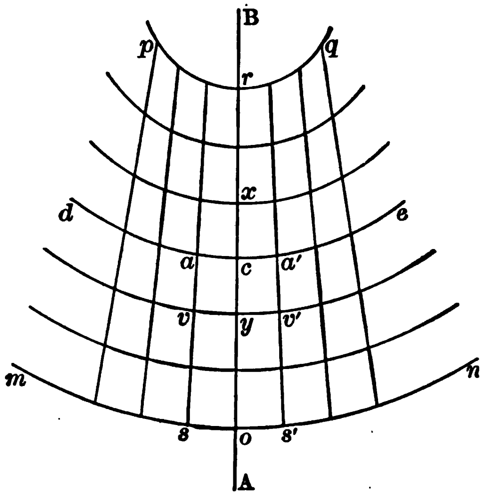


FIG. 109.

In the same way, the representations of the two meridians next to these may be constructed by laying off on the same arcs from a, v, s, etc., distances each equal to ca, yv, os, etc., and drawing lines through the points thus determined, and so on, until the representations of the extreme meridians of the portion to be mapped are drawn.

In this map the scale along the central meridian and parallels will be accurate. In other directions, when the map does not represent a very extended portion of the sphere, a very near approximation to accuracy is made.

If c is a point of the equator, the tangent cone becomes a cylinder, the arcs become straight lines, and the meridians become parallel.

The Polyconic Method

242. In the polyconic method, by which the elegant and accurate maps of the United States Coast Survey are constructed, the central meridian and parallel are represented as in the preceding article. The representations of the other parallels are each described, through the proper point of division, with a radius equal to the tangent of its polar distance, thus giving the development of the lines of contact of so many cones tangent to the sphere. The length of each degree of longitude is then laid off as in the former method, and the representations of the meridians drawn.

In this method, also, the scale along the central meridian and parallels is accurate, and in other directions very nearly so. This has the advantage that the representations of the meridians and parallels are perpendicular to each other as in space, which is not the case in Flamstead's method.

243. When very small areas are mapped, this method is thus modified in the Coast Survey Office. The same process is used as above to construct the representations of the meridians with accuracy. Then commencing with the central parallel, the distance cx, Fig. 109, between it and the consecutive one, as measured on AB, is laid off on each meridian in both directions, and through the extremities lines are drawn to represent the consecutive parallels. Then from these the same distances are laid off for the next, and so on until all are constructed.

The first set of parallels described as in the preceding article, which should be in pencil, are then erased.

By this method equal meridian distances are everywhere included between the parallels, and the scale is accurate only in the direction of the meridians and central parallel. This is called the *equidistant polyconic method.*

244. When the polar distance of the parallel is much greater

than 45°, the practical construction of its representation becomes difficult, as its center will be so far distant. In such case, for both the polyconic method and that of Flamstead, tables are carefully computed giving the rectilinear coördinates of the points of the representations of the parallels, for each minute of latitude and longitude, and these representations can then be accurately constructed by points. The tables thus computed in the United States Coast Survey office are very much extended and of great value.

PART III

SHADES AND SHADOWS

Preliminary Definitions

245. Effects of light and shade. To represent a body with accuracy, it is not only necessary that the drawing should give the representations of the details of its form, but also of its colors, whether natural or artificial, or the effect of light and shade.

Different portions of the same body will appear lighter or darker according as the light falls directly upon it or is excluded from it by itself or some other body. A simple application of the elementary principles previously deduced will enable us to limit and represent these portions, and constitutes that part of the subject called *Shades and Shadows.*

246. Rays of light. It is a principle of Optics that the effect of light, when in the same medium and unobstructed, is transmitted from each point of a luminous body in every direction, along straight lines. These straight lines are called ***rays of light.*** ***Any straight line,*** therefore, ***drawn from a point of a luminous body*** will be regarded as a ray of light.

The sun is the luminous body which is the principal source of light. It is at so great a distance, that rays drawn from any of its points to an object on the earth's surface may, without material error, be regarded as parallel. In the construction of problems, in this part of our subject, they will be so taken.

247. Indefinite shadow. Let SR, Fig. 110, indicate the direction of the parallel rays of light, coming from the source and

falling on the opaque body B; and let *n-mlop-q* be a cylinder whose rectilinear elements are rays of light parallel to SR, enveloping and touching this body.

It is evident that all light, coming directly from the source, will be excluded from that part of the interior of this cylinder which is behind B. This *part of space from which the light is thus excluded by the opaque body, is the indefinite shadow of the*

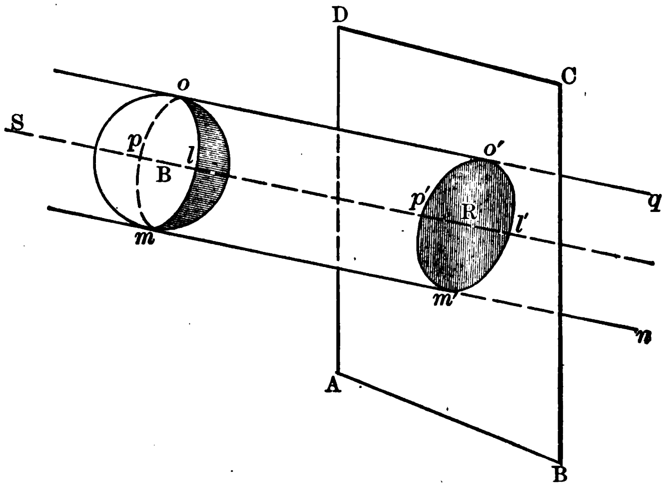


FIG. 110.

body; and any object within this portion of the cylinder is *in the shadow or has a shadow cast upon it.*

Shade. The line of contact *mlop* separates the surface of the opaque body into two parts. That part which is towards the source of light, and on which the rays fall directly, is the *illuminated part;* and that part opposite the source of light from which the rays are excluded by the body itself, is the *shade of the body.*

Line of shade. This line of contact of the tangent cylinder of rays, thus *separating the illuminated part from the shade,* is the *line of shade.*

Plane of rays. Any plane tangent to this cylinder of rays will also be tangent to the opaque body at some point of this line of contact (Art. 144); and, conversely, every plane tangent to the opaque body at a point of the line of shade will be tangent to this cylinder, and contain a rectilinear element (Art. 83), or ray of light, and thus be a *plane of rays.* Hence,

To obtain points of the line of shade on any opaque body, pass planes of rays tangent to the body, and find their points of contact.

248. Shadow cast on a surface. If the cylinder *n-mlop-q* be intersected by any surface, as the plane ABCD, that part *m'l'o'p'* of this plane, which is in the shadow, is the shadow of B on this plane. That is, *the shadow of an opaque body on a surface is that part of the surface* from which the rays of light are excluded by the interposition of this body between it and the source of light.

Line of shadow. The bounding line of this shadow, or the *line which separates* the shadow on a surface from the illuminated part of that surface, is the *line of shadow.* It is also the line of intersection of the cylinder of rays which envelops the opaque body, and the surface on which the shadow is cast.

249. Lines of shade and shadow on a body bounded by planes. When the opaque body is bounded by planes, the cylinder of rays, by which its shadow is determined, will be changed into several *planes of rays*, which will include the indefinite shadow; and the line of shade will be made up of straight lines which, though not lines of mathematical tangency, are the outer lines in which these planes touch the opaque body, and still *separate the illuminated part from the shade.*

The line of shadow in this case is also made up of the lines of intersection of these planes with the surface on which the shadow is cast, and still separates the illuminated part of this surface from the shadow.

Shadows of Points and Lines

250. Shadow of a point. The indefinite shadow of a point may be regarded as that part of a ray of light, drawn through it, which lies in the direction opposite to that of the source of light, and the point in which this ray pierces any surface is the *shadow* of the point *on that surface.*

251. The shadow of a straight line will then be determined by drawing through its different points rays of light. These form a *plane of rays*, and the intersection of this plane with any surface is the *shadow of the line on that surface.*

To determine whether a given plane is a plane of rays, it is only necessary to ascertain whether *it contains a ray of light.* This may be done by drawing through any one of its points a ray. If this pierces the planes of projection in the traces of the given plane, it is a plane of rays (Prop. XXI, Art. 14).

252. The shadow of a straight line on a plane may be constructed by finding the shadows cast by any two of its points on the plane (Art. 250), and joining these by a straight line, or by joining the shadow of any one of its points with the point in which the line pierces the plane (Art. 42).

If the straight line be parallel to the plane, its shadow on the plane will be parallel to the line itself, and the shadow of a definite portion of it will be equal to this portion (Prop. XIV, Art. 14).

If the line is a ray of light, its shadow on any surface will be a point.

To find the shadow of a given straight line on a given plane. Let MN, Fig. 111, be a limited straight line, and T the plane on which its shadow is required, MR indicating the direction of the rays of light.

Through the extremities M and N draw the rays RM and NS. They pierce the plane T in R and S (Art. 42), which are

points of the shadow (Art. 250). Join these points by RS; it will be the required shadow.

If the shadow be cast on the horizontal plane, the rays through the extremities M and N, Fig. 112, pierce H at m_1 and n_1, and m_1n_1 is the shadow.

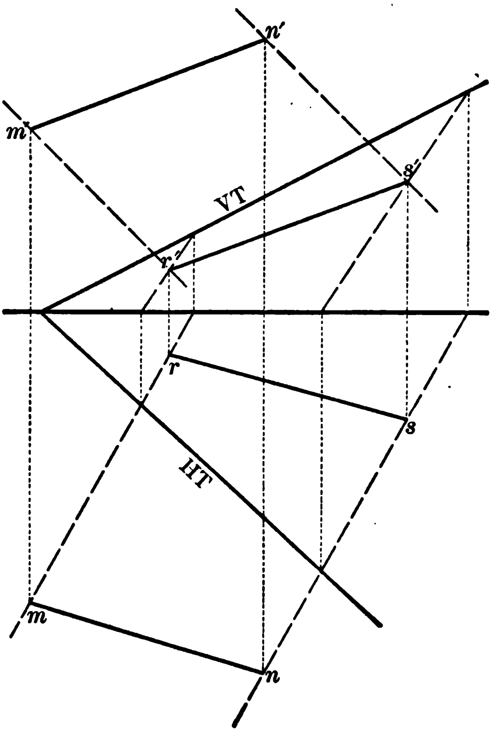


FIG. 111.

If the shadow be cast on the vertical plane, these rays pierce V at m_1 and n_1, Fig. 113, and m_1n_1 is the shadow.

If MN is parallel to either plane, then RS in Fig. 111, and m_1n_1 in Figs. 112 and 113, will be equal to MN.

If straight lines are parallel, their shadows on a plane are parallel, since they are the intersections of parallel planes of rays by this given plane.

253. The shadow of a curve will be determined by passing through it a cylinder of rays. The *intersection of this cylinder with any surface* will be the shadow of the curve *on* that surface.

To find the shadow cast by any line in space upon any other line. The line of shadow on a surface (Art. 248) will always be the shadow of the line of shade, and if through any point of the line of shadow a ray be drawn to the source of light,

it will intersect the line of shade in a point which casts the assumed point of shadow.

From this it follows that the point of shadow cast by any line in space upon any other line may be found by constructing the shadows of both lines on a plane (say the horizontal plane), and drawing through the point of intersection of these shadows a ray; it will intersect the second line in the point of shadow cast upon it, by the point in which it intersects the first.

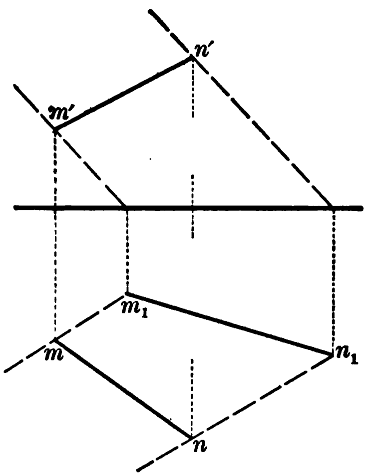


FIG. 112.

254. To find points of the shadow cast by a line upon a curved surface. Assume an oblique cylinder, and any curved line outside the surface, and let it be required to find the shadow of the curved line upon the cylinder. This may easily be done by an application of the foregoing principle. Find the shadow of the curve on H (under the supposition that the rays are unobstructed), and also the shadows of several elements of the cylinder. From the points where the shadow of the curve cuts the shadows of the several elements draw rays of light. The points where these rays intersect the corresponding elements of the cylinder will be points on the required shadow. Let the student make the construction.

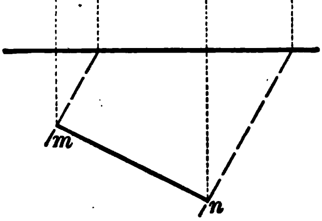


FIG. 113.

255. Shadows of two tangent lines. If two lines are tangent in space, their shadows, on any surface, will be tangent at the point of shadow cast by the given point of contact. For the cylinders of rays through the lines will be tangent along the ray passing through the point of contact (Art. 145); hence their intersections by any surface will be tangent at the point where this ray pierces the surface. These intersections are the shadows of the lines.

256. The shadow of a curve of single curvature, on a plane to which it is parallel, will be an *equal curve*, since each of its elements will cast a parallel and equal element of the shadow (Art. 252).

If the plane of the curve be a plane of rays, its shadow on a plane will evidently be a straight line.

The shadow of the circumference of a circle, on a plane to which it is parallel, will be an *equal circumference*, whose center is the shadow cast by the given center.

Also, when the plane on which the shadow is cast makes a *subcontrary section* in the cylinder of rays through the circumference (Art. 215), the shadow will be an equal circumference.

In all other positions, when its plane is not a plane of rays, its shadow will be an ellipse (Art. 106), *and any two diameters of the circle perpendicular to each other* will cast *conjugate diameters of the ellipse* of shadow. For if at the extremities of either diameter tangents be drawn, they will be parallel to the other diameter, and their shadow parallel to the shadow of this diameter (Art. 252). But the shadows of the two tangents are tangent to the shadows of the circle at the extremities of the shadow of the first diameter (Art. 255); hence the shadow of the second diameter is parallel to the tangents at the extremities of the shadow of the first. These shadows are therefore conjugate diameters of the ellipse, and with them the ellipse may be constructed by the following method.

257. Given any pair of conjugate diameters of an ellipse, to construct the curve. Two oblique diameters of an ellipse are said to be conjugate when each is parallel to the tangents at the extremities of the other. Let AB and CD, Fig. 114, be

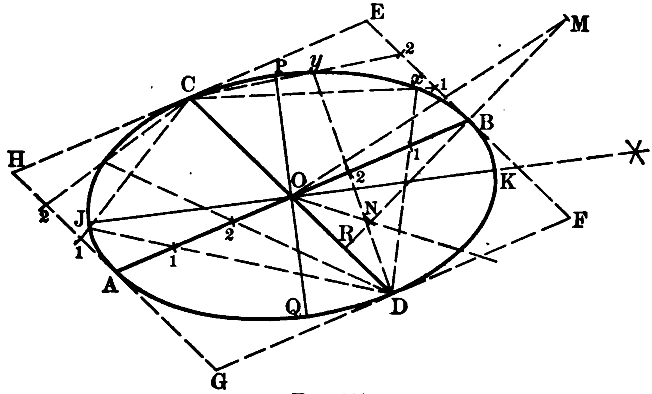


FIG. 114.

two given conjugate diameters. At the extremities of each, draw lines parallel to the other. These will form a parallelogram EFGH, and be tangent to the ellipse at the points A, B, C, D, respectively. Divide BO into a convenient number of equal parts, 1, 2, etc. Divide BE into the same number of equal parts. Draw lines from C and D through the equal divisions as shown, and determine the points x, y, etc., on the ellipse. In the same manner find points for the other three parts of the curve.

258. Given a pair of conjugate diameters of an ellipse, to determine the major and minor axes. From B, Fig. 114, the end of the long diameter, draw a perpendicular, BR, to the direction of the shorter, and upon it lay off BM and BN each equal to OD, half the shorter diameter. Bisect the angle MON, thus obtaining the direction KO, of the major axis. The length of the major axis, JK, is OM + ON. The length of the minor axis, PQ, is OM – ON.

Practical Problems

259. Problem 76. **To construct the shadow of a rectangular pillar on the planes of projection.**

Let *mnlo*, Fig. 115, be the horizontal and $m'n'p'q'$ the vertical projection of the pillar, and let $(nn_1, n'r)$ indicate the direction of the rays of light.

The upper base vertically projected in $m'n'$, and the two vertical faces horizontally projected in *mn* and *mo*, are evidently illuminated, and together form the illuminated part of the pillar; and the edges $(n, p'n')$, (nl, n'), $(lo, n'm')$, and $(o, q'm')$ separate the illuminated part from the shade and make up the line of shade (Art. 249).

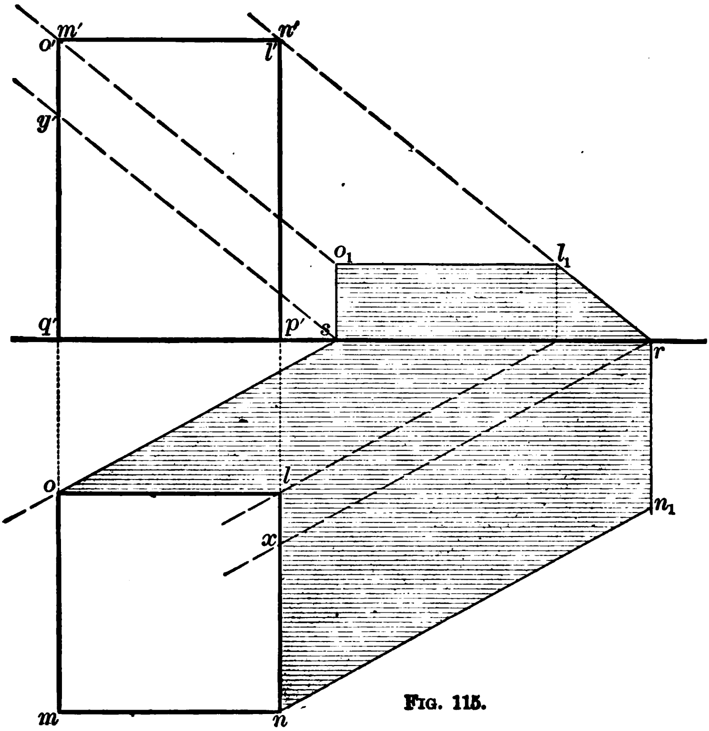


Fig. 115.

Since n is in the horizontal plane and n_1 the shadow of N, nn_1 is the shadow of $(n, p'n')$ on H (Art. 252). The plane of rays through (nl, n') is perpendicular to V, and n_1r, parallel to nl, is its horizontal and rl_1 its vertical trace; hence n_1r is that part of the shadow of (nl, n') which is on H; and rl_1, limited by the ray through (ln'), that part on V. The ray (rx, rn') intersects (nl, n') in (x, n'), and nx is the horizontal projection of that part which casts the equal shadow n_1r (Art. 252), and xl of that part which casts rl_1.

l_1o_1 is the shadow of $(ol, m'n')$ on V, parallel and equal to itself.

The plane of rays through $(o, q'm')$ is perpendicular to H, and os is its horizontal and so_1 its vertical trace; and os is the shadow of $(o, q'm')$ on H, and so_1 its shadow on V. $q'y'$ is the vertical projection of the part which casts the shadow os, and $y'm'$ of that part which casts its equal so_1 (Art. 252).

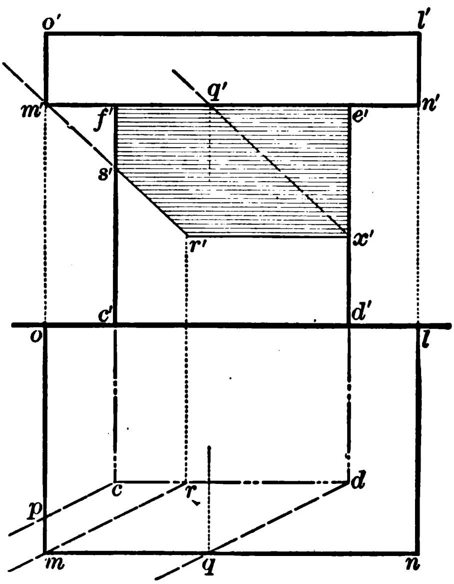


FIG. 116.

The line of shadow is thus the broken line $nn_1r - - o_1 - - o$, and the portion of the planes within this line is the required shadow, and should be darkened as in the drawing.

260. PROBLEM 77. **To construct the shadow of a rectangular abacus on the faces of a rectangular pillar.**

Let $mnlo$, Fig. 116, be the horizontal, and $m'n'l'o'$ the vertical projection of the abacus, and $cdd'c'$ the horizontal, and $c'd'e'f'$

the vertical projection of the pillar; MR indicating the direction of the rays.

The lines of shade on the abacus, which cast the required shadows, are evidently the two edges (mo, m') and MN.

The plane of rays through (mo, m') is perpendicular to V, and intersects the side face, horizontally projected in cc', in a straight line perpendicular to V at s', which is the shadow on this face. $m'r'$ is the vertical trace of this plane. The ray MR, through M, pierces the front face of the pillar in R (Art. 42), the shadow of M, and (cr, $s'r'$) is the shadow of the part (pm, m') on this face.

MN being parallel to this face, its shadow on this face will be parallel to itself, and pass through R; hence (rd, $r'x'$) is the required shadow cast by its equal MQ.

261. Problem 78. **To construct the shadow of an upright cross upon the horizontal plane and upon itself.**

Let $mnop$, Fig. 117, be the horizontal, and $c'n'r'g'f'$ the vertical projection of the cross.

cc_1 is the shadow of the edge (c, $c'd'$), c_1 being the shadow of (c, d'); c_1n_1 is the shadow of its equal (cn, $d'n'$) (Art. 252); n_1h_1 of the edge (n, $n'h'$); h_1o_1 of its equal (no, h'); o_1y_1 of (oy, $h'x'$); xy of a part of (cd, r'), on the face of the cross vertically projected in $l'h'$; y_1r_1 is the shadow of the remaining part of (cd, r') on H. cx is the shadow of (c, $l'r'$) on the face vertically projected in $l'h'$, this being the intersection of a vertical plane of rays through (c, $l'r'$) with this face. r_1q_1 is the shadow of its equal (de, $r'q'$); q_1k_1 of (e, $k'q'$); k_1p_1 of (pe, $k'g'$); p_1m_1 of (p, $m'g'$); and m_1t of a part of (pm, m').

All within the broken line thus determined on H is darkened, as also the part $cxyd$, the horizontal projection of the shadow cast by the upper part of the cross on the face vertically projected in $l'h'$.

As the solution of this problem consists mainly in finding the

horizontal piercing points of lines, it should present no difficulty beyond that of determining which of the lines cast shadows.

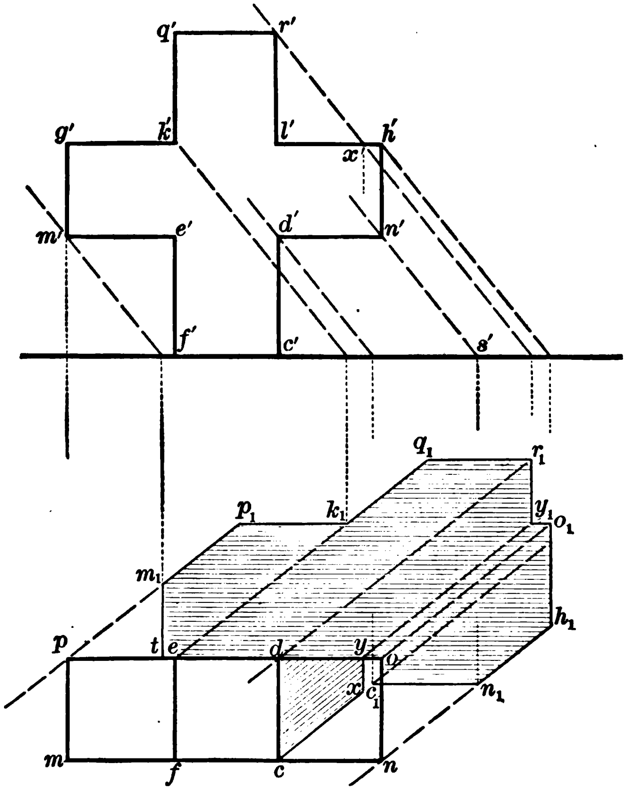


FIG. 117.

262. PROBLEM 79. **To construct the shade of a cylindrical column, and of its cylindrical abacus, and the shadow of the abacus on the column.**

Let *mlo*, Fig. 118, be the horizontal, and *m'o'h'n'* the vertical projection of the abacus, *cde* the horizontal, and *cee'g'* the vertical projection of the column.

Pass two planes of rays tangent to the column. Since each contains a rectilinear element, they will be perpendicular to H;

and *ld* and *kf*, parallel to the horizontal projection of the ray of light, will be their horizontal traces (Art. 93); and (d, $z'd'$) and (f, $f'r'$) the elements of contact, which are indefinite lines, *or elements of shade* (Art. 247).

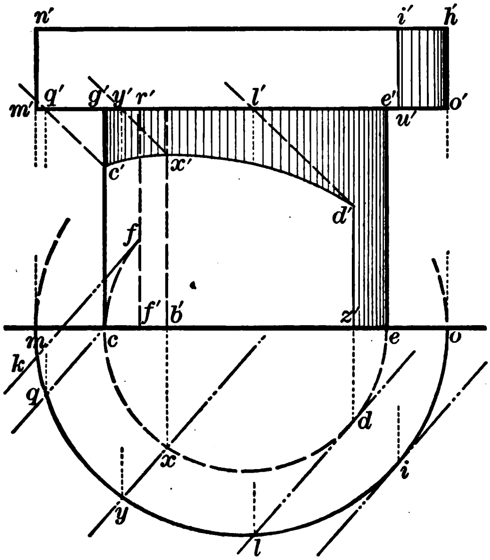


FIG. 118.

In the same way the elements of shade (i, $u'i'$), etc., on the abacus, are determined.

The line of shadow on the column will be cast by a portion of the lower circumference of the abacus toward the source of light. To determine any point of it, pass a vertical plane of rays, as that whose horizontal trace is yx. It intersects the column in a rectilinear element (x, $b'x'$), and the circumference in a point Y. Through this point draw the ray YX. It intersects the element in X, a point of the required shadow (Art. 250). In the same way any number of points may be found. C is the shadow of Q, and D of L, and $c'x'd'$ is the vertical, and cxd the horizontal projection of that part of the curve of shadow which can be seen.

That part of the shade on the abacus and column, and of the shadow on the column, which can be seen, is darkened in the drawing.

263. PROBLEM 80. **To construct the shade of an oblique cone and its shadow on the horizontal plane.**

Let the cone be given as in Fig. 119, S being its vertex, and the base resting in H.

If two planes of rays be passed tangent to the cone, the elements of contact will be the *elements of shade* (Art. 247). Since

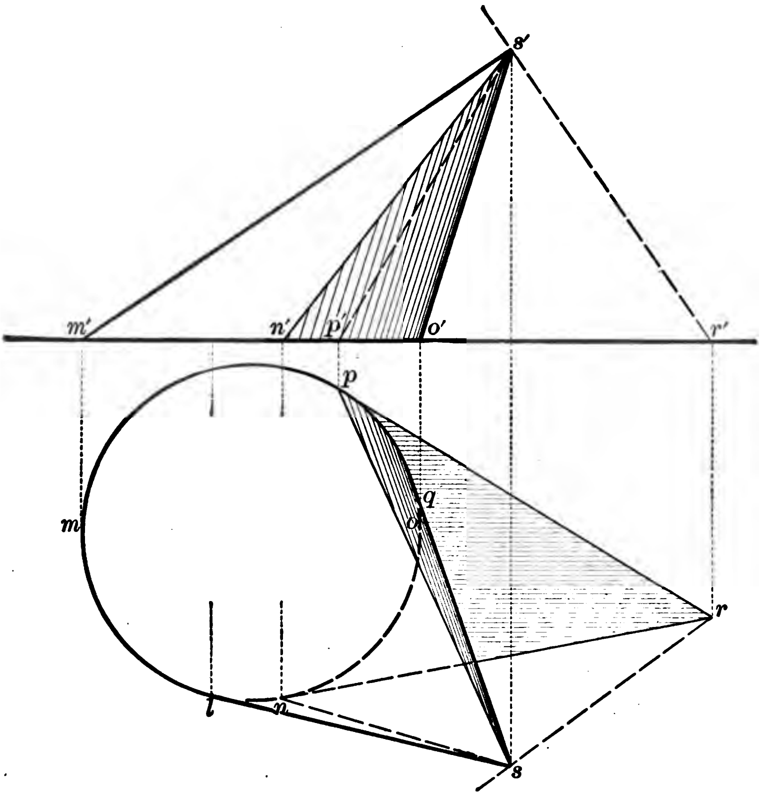


FIG. 119.

these planes must pass through S (Art. 100), they must contain the ray of light SR. This pierces the plane of the base at *r*, and *rn* and *rp* are the intersections of these tangent planes with the plane of the base, and SN and SP are the elements of contact (Art. 100).

These elements cast the shadows *nr* and *pr*, and together with the shadow of the curve of the base (in this case coincident with the curve of the base), limit the shadow of the cone on H.

In looking upon H, the shade between the elements SP and SQ only is seen; and in looking upon V, that between SN and SO.

Let the student assume the cone with its base resting in a plane parallel to H and one inch above it. Find the elements of shade and the shadow on H. How will the solution differ from that in Fig. 119?

264. PROBLEM 81. **To construct the shade of an oblique cylinder, and its shadow on the horizontal plane.**

If two planes be passed tangent to the cylinder and parallel to the rays of light (Art. 95), their horizontal traces will be the bounding lines of the shadow cast upon H, the vertical traces will be the bounding lines of the shadow on V, and the elements of contact will be the lines of shade. Let the student make the construction.

265. PROBLEM 82 *a*. **To construct the shade of a circular ring, and its shadow on the horizontal plane and on its interior surface.**

Let *gfb*, Fig. 120, be the horizontal projection of the ring, and $(c, d'c')$ its axis.

The two planes of rays whose horizontal traces are ll_1 and kk_1, determine the two *elements of shade* $(l, n'l')$ and $(k, r'k')$.

These elements cast the shadows ll_1 and kk_1 on H (Art. 252).

The semicircumference of the upper base GYB casts its shadow on the interior of the ring, and the semicircumference KOL on the horizontal plane, without the ring.

The shadow of the latter is the equal semicircumference $k_1o_1l_1$ whose center is c_1 (Art. 256), and this, with the lines ll_1 and kk_1, limits the shadow of the cylinder on H.

To determine the shadow on the interior surface, pass any ver-

tical plane of rays, as that whose horizontal trace is *yx*. It intersects the cylinder in the element $(x, w'x')$ and the semicir-

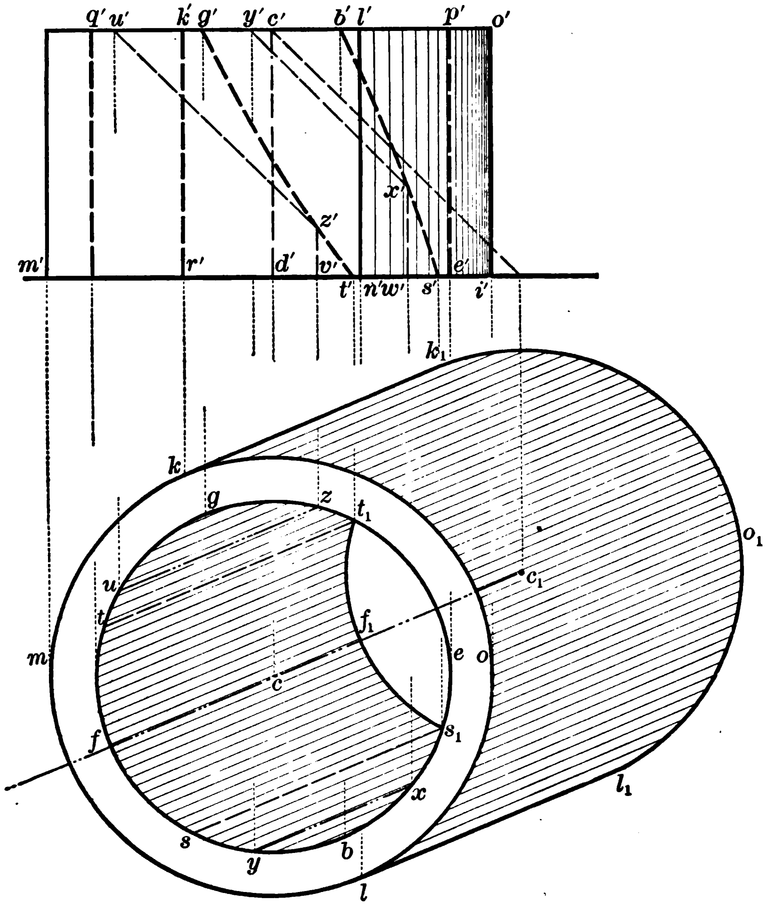


FIG. 120.

cumference in the point Y. The ray of light YX intersects $(x, w'x')$ in X, a point of the required shadow. In the same way any number of points may be determined.

The shadow upon any rectilinear element may be found by

using an auxiliary plane which shall contain this element. **Thus** z is the shadow of U on the element (z, $v'z'$), s_1 of S, and t_1 of T.

This shadow evidently begins at B and G.

The direction of the rays is so taken that a part of this semicircumference casts its shadow on H, within the cylinder. This part is horizontally projected in sft, and its shadow is the equal arc $s_1 f_1 t_1$ with center at c_1.

Should this shadow not reach H, its *lowest point* will be obtained by using as an auxiliary plane that one which contains the axis; and the point in which the vertical projection of the curve of shadow is tangent to $e'p'$, by using the plane which contains the element (e, $e'p'$).

That part of the shade between the elements (l, $n'l'$) and (o, $o'i'$) only can be seen when looking on V, and is shaded in the drawing.

266. PROBLEM 82 *b*. **Same as the preceding, but with the ring resting in an oblique plane.**

Let the student construct his own figure, assuming a circular ring of 2 inches diameter, 1 inch in height, and of material of no appreciable thickness (*e.g.* a paper band). Let it rest in a plane S, whose horizontal trace is perpendicular to the ground line, and whose vertical trace is inclined 30° to the ground line. Let the center of the lower circle of the ring be $1\frac{1}{8}$ inches from H and $1\frac{3}{8}$ inches from V. The projections of the rays of light will each be inclined 45° to the ground line. The projections of the ring may be constructed by the method of Art. 45.

The line of shade may be found by passing a plane of rays tangent to the cylinder and drawing the element of contact. Choose any point in space, as M, and through it pass a ray of light MT, and a line MV parallel to the elements of the ring. The plane of rays T, determined by these two lines, cuts S, the plane of the lower circle of the ring, in the line AB. Parallel to this line draw CD and EF tangent to the lower circle of the

ring. They are the lines cut from S by planes of rays tangent to the cylinder. The elements drawn from the points of tangency (found as in Art. 67) are the lines of shade required.

The shadow cast by the upper circle upon the interior surface may be determined by passing other planes of rays parallel to the elements. Each will cut from the plane of the upper circle a line such as GK (parallel to AB) and from the cylinder an element KN. A ray through G will be contained in this plane, and the shadow of G will fall upon the element at N. Similarly for other points, which when found will be joined by a smooth curve.

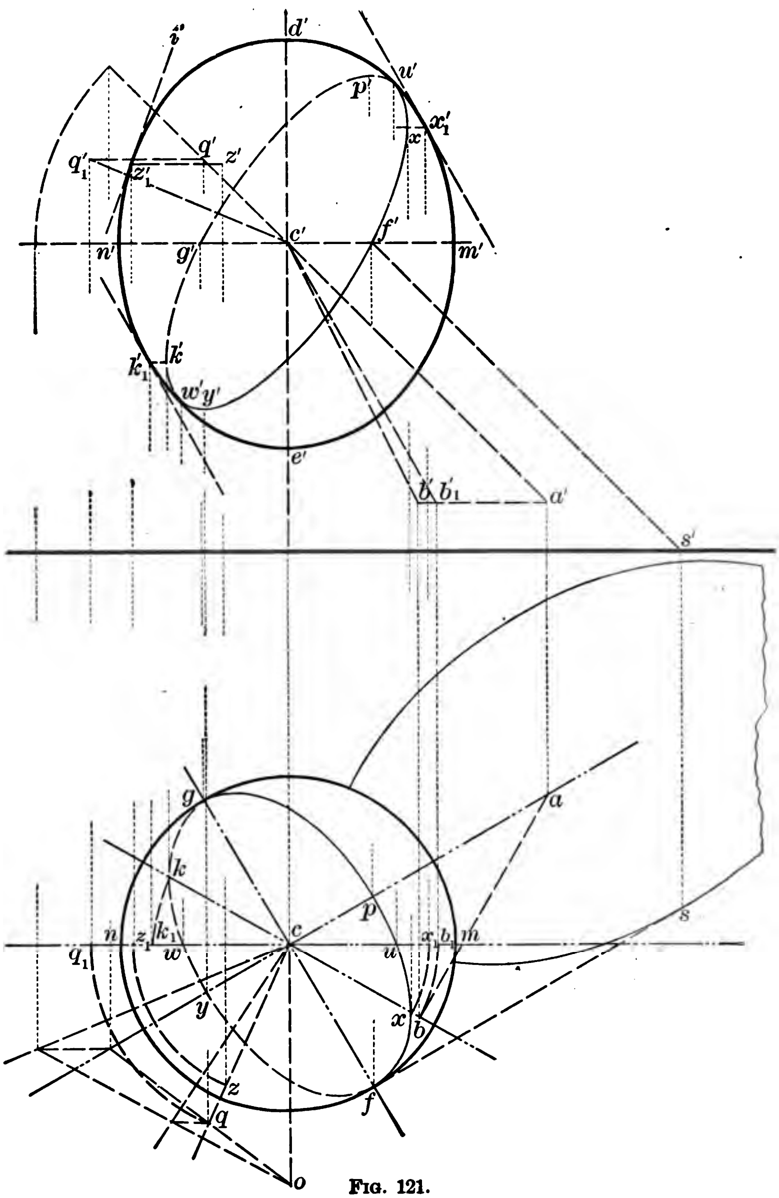
The shadow cast by the ring upon H may be found by the method of Arts. 256 and 257.

267. PROBLEM 83. **To construct the line of shade on any surface of revolution.**

Analysis. If any meridian plane be assumed, and a ray of light be projected upon it, the projecting plane will be a plane of rays perpendicular to the meridian plane, and the projected ray will be the trace of the plane of rays on the meridian plane. If now a line be drawn tangent to the meridian curve and parallel to this projected ray, it will be the trace of another plane of rays parallel to the first, and hence perpendicular to the meridian plane. This second plane of rays must then be tangent to the surface of revolution (Art. 175) at the point of tangency of the line with the meridian curve. This point must then be a point on the line of shade (Art. 247).

If now another meridian plane be assumed, and a ray of light be projected upon it, and parallels be drawn to the meridian curve, the points of tangency will give other points on the line of shade.

Construction. Let the surface of revolution be an ellipsoid, Fig. 121, and let CA be a ray of light taken, for convenience, through any point C of the axis. Assume any meridian plane,



i'
d'
u'
p'
x
x'_1
q'_1
q'
z'
z'_1
c'
f'
n'
g'
m'
k'
k'_1
w'y'
e'
b'
b'_1
a'
s'
g
k
a
p
n
z_1
k_1
c
u
x_1
b_1
m
s
q_1
w
y
x
b
z
f
q
o

FIG. 121.

as the one whose horizontal trace is *kb*, and project the ray CA upon it. B is the projection of A upon the plane, and C is its own projection. CB is then the projection of the ray.

Revolve the meridian plane parallel to V. CB takes the position (cb_1, $c'b'_1$), and the meridian curve is vertically projected in the ellipse $m'd'n'e'$. Draw two lines parallel to $c'b'_1$ and tangent to the ellipse at the points x'_1 and k'_1, found as in Art. 67. These points are horizontally projected in x_1 and k_1, and are the revolved positions of points on the line of shade.

Revolving the meridian plane to its primitive position, the required points are found at xx' and kk'.

In like manner any number of points may be found on the line of shade by assuming meridian planes in various positions. The meridian plane parallel to V will give the points U and W, whose vertical projections show the points of tangency between the vertical projection of the line of shade and the contour line of the surface. The meridian plane through the ray of light CA will give the highest and lowest points P and Y, or the points on the axis of symmetry.

The shadow cast by the curve of shade upon H is found by passing rays of light through various points of the curve of shade, and finding their horizontal piercing points.

268. If the surface of revolution is a sphere, the construction of the curve of shade is simplified, as it then becomes the circumference of a great circle perpendicular to CA, and may be constructed by the methods of Arts. 44 and 45.

Brilliant Points

269. In looking upon the illuminated part of a curved surface it will be observed that, in general, one or more points appear much more brilliant than the others. This is due to the fact that the *ray of light incident at each of these points is reflected immediately to the point of sight.*

These points are called *brilliant points;* and since it is a principle of Optics that the incident and reflected rays, at any point of a surface, lie in the same normal plane on opposite sides of the normal at this point, making equal angles with it, it follows that at a brilliant point the ray of light and the straight line drawn to the point of sight must fulfill these conditions.

To find the brilliant point on any surface. When the point of sight is at an infinite distance, as in the orthographic projection, the brilliant point may be constructed thus: Through any point draw a ray of light and a straight line to the point of sight. These will be parallel respectively to the corresponding lines at the required point and make the same angle. Bisect the angle formed by these auxiliary lines, as in Art. 39. The bisecting line will be parallel to the line which bisects the parallel and equal angle at the brilliant point. Hence, *perpendicular to this bisecting line, and tangent to the surface,* pass a plane. Its point of contact *will be the brilliant point;* for a line through this point parallel to the bisecting line will be a normal to the surface, and will bisect the angle formed by a ray of light and a straight line drawn from this point to the point of sight.

270. PROBLEM 84. **To find the brilliant point on any surface of revolution.**

Let the surface be an ellipsoid given as in Fig. 121. Through any point of the axis, as C, draw the ray of light CA and the line (co, c') to the point of sight in front of the vertical plane. Bisect their included angle, as in Art. 39, by the line CQ. To determine a plane perpendicular to this line and tangent to the surface (Art. 269), through it pass the meridian plane of which cq is the horizontal trace. This plane cuts from the surface a meridian curve, and from the required tangent plane a straight line tangent to this curve at the required point and perpendicular to CQ.

Revolve this plane about $(c, c'd')$ until it becomes parallel to

V. The meridian curve will be vertically projected in $e'n'd'm'$, and CQ in $c'q'_1$ (Art. 26). Tangent to $e'm'd'$ and perpendicular to $c'q'_1$ draw $i'z'_1$; z'_1 is the vertical projection of the revolved position of the required point horizontally projected at z_1, when the plane is revolved to its true position at z. Z is then the required brilliant point.

271. PROBLEM 85. **To construct the shade and shadow of an upright screw.**

Let exf, Fig. 122, be the circular base of a right cylinder whose axis is oo_2, and let $em'e'$ be an isosceles triangle in the plane of the element ee' and of the axis, its base coincident with the element. If this triangle be moved about the cylinder, its plane always containing the axis, with a uniform angular motion, and at the same time with a uniform motion in the direction of the axis, so that after passing around once it will occupy the position $e'm'_1e'_1$, it will generate a volume called the *thread of a screw* (Art. 153).

The cylinder is the *cylinder of the screw.*

It is evident that the two sides $m'e$ and $m'e'$ will each generate a portion of a helicoid (Art. 150), the side $m'e'$ generating the *upper surface* of the thread, and $m'e$ the *lower surface.*

The point m' generates the *outer helix* of the thread, and the point e the *inner helix.*

The curve of shade on the lower surface of the thread may be constructed by passing planes of rays tangent to this surface and joining the points of contact by a line (Art. 247).

To find the point of this curve on the *inner helix* pass a plane tangent to the lower helicoid at F; ge_2 is the horizontal trace of this plane, the distance fe_2 being equal to the rectified arc fxe (Art. 154).

It is a property of the helicoid that all planes tangent to the surface *at points of the same helix*, make the same angle with the axis, or with the horizontal plane when taken perpendicular to

this axis. For each of these planes is determined by a tangent to the helix and the rectilinear element through the point of contact; and these lines at all points of the helix make the same angle with the axis (Arts. 73 and 151) and with each other.

Hence, if through the axis we pass a plane perpendicular to the tangent plane at F, it will cut from this plane a straight line, intersecting oo_2 at o', horizontally projected in oi, and making with H the same angle as that made by a plane tangent at any point of the helix, and with the rectilinear element passing through the point of contact an angle which is the same for all these planes; hence the angle iof will be the same for all these planes.

If this line be now revolved about oo_2, it will generate a cone whose rectilinear elements make the same angle with H as the tangent plane. If a plane of rays be passed tangent to this cone, it will be parallel to the plane of rays tangent to the surface at a point on the inner helix. kl, tangent to the circle generated by oi, is the horizontal trace of this plane (Art. 101), and ol is the horizontal projection of the line cut from the parallel tangent plane of rays by a perpendicular plane through oo_2. If, then, we make the angle lox equal to iof, ox will be the horizontal projection of the element containing the required point of contact, and X will be the point.

In the same way, the point Y on the outer helix may be obtained by first passing a plane tangent to the lower surface at M. eq is its horizontal trace, mq being equal to the rectified semicircumference myn, and eop the constant angle. A tangent from t to the circle generated by op is the horizontal trace of the parallel plane of rays, and oy, making with the straight line joining o and the point of contact of this tangent, an angle equal to eop, the horizontal projection of the element containing the point of contact, and Y the required point.

Intermediate points of the curve of shade may be determined

by constructing, as above, the points of contact of tangent planes of rays on intermediate helices. Thus, pass a plane tangent to the lower helicoid at W, midway between F and N; wbz is the horizontal projection of the helix passing through this point; fb_1 the horizontal projection of the intersection of the tangent plane with the horizontal plane through f', wb_1 being equal to the rectified arc wb. Since this intersection is parallel to the horizontal trace of the tangent plane, ou will be the horizontal projection of the line cut from the tangent plane by the perpendicular plane through oo_2, and fou the constant angle. us is the horizontal projection of the circle cut from the auxiliary cone by the horizontal plane through $f'm'$, and ts the horizontal projection of the intersection of the tangent plane of rays with the same plane; oz making with the straight line joining o and s, an angle equal to fou, is then the horizontal projection of the element containing the point of contact, o_5v' (Art. 151) its vertical projection, and Z the required point.

As the surface, of which $e'm'_1n'_1f'_1$ is the vertical projection, is in all respects identical with that of which $em'n'f'$ is the projection, $y'_1x'_1$ will be the vertical projection of the curve of shade on the first surface, its horizontal projection being yx.

To construct the shadow cast by this curve of shade on the surface of the thread below it, construct the shadow of $(xy, x'_1y'_1)$ on the horizontal plane through $n'e'$, under the supposition that the rays are unobstructed. x_2y_2 is the horizontal projection of this shadow (Art. 253). Then assume any element, as $(do, d'o_1)$, on which it is supposed the shadow will fall, and construct its shadow on the same plane. h_1r_1 is the horizontal projection of this shadow. Through the point in which these shadows intersect, horizontally projected at r_1, draw a ray of light, intersecting the element in R, which is the shadow of the curve on the element (Art. 253). The curve evidently begins at (x, x'_1) and is vertically projected in x'_1r'.

The shadow cast by the helix (mn, $m'_1n'_1$) **on the surface of the thread** may be constructed in the same way by first constructing its shadow on the same horizontal plane as above, and the shadow of an assumed element ($o\delta$, $o_2\delta'$) on the same plane, and from their point of intersection horizontally projected at δ_1, drawing a ray; (δ, δ') will be a point of the required shadow.

The curve of shadow on the surface, whose vertical projection is enf', will be equal to the curve whose vertical projection is $x'_1r'\delta'$, and may be drawn as in the figure.

Since the plane of rays through Y is tangent both to the line of shade YX and to the helix YDN, the shadows of these curves on the horizontal plane through $e'n'$ will be tangent at the point y_2, the shadow of Y upon this plane.

272. The brilliant point on the upper surface of the thread may be constructed by bisecting the angle formed by a ray of light and a line drawn to the point of sight (Art. 269), and then passing a plane perpendicular to the bisecting line and tangent to the surface (as in Art. 155), and finding its point of contact.

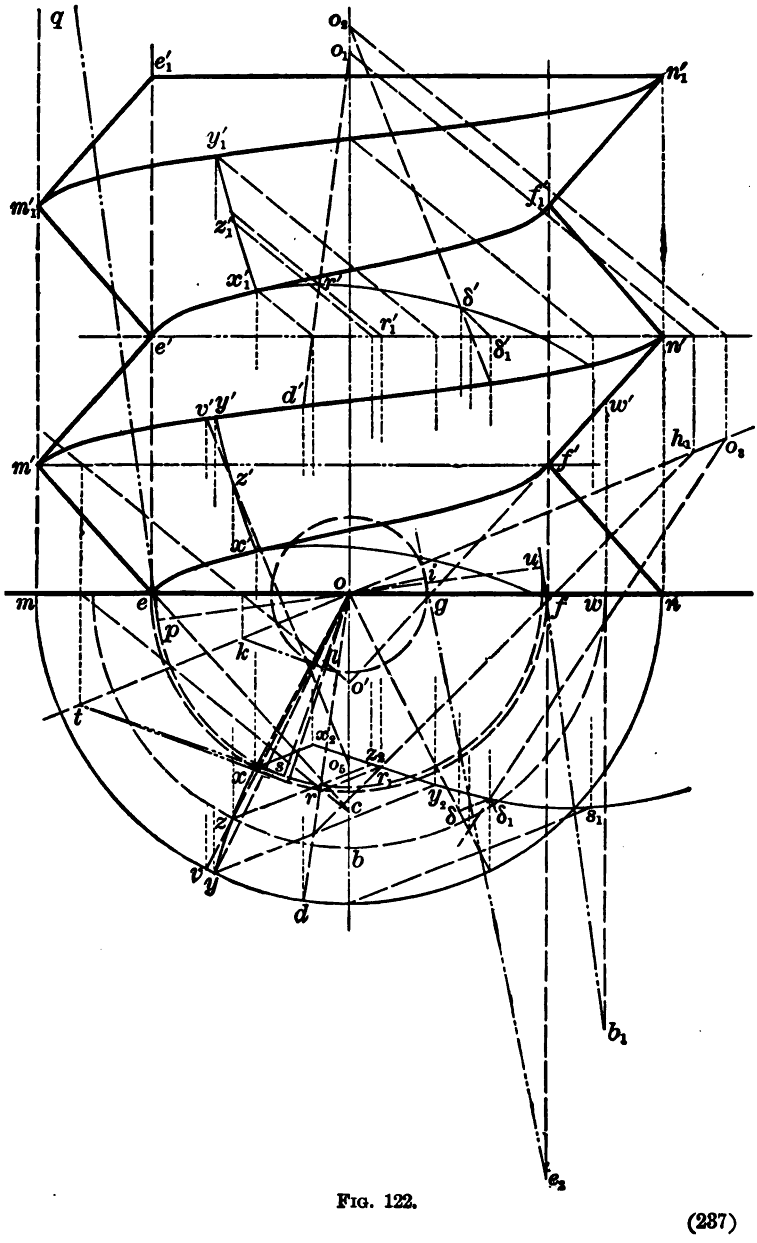

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FIG. 122.

PART IV

LINEAR PERSPECTIVE

Preliminary Definitions and Principles

273. It has been observed (Art. 213) that the orthographic projections can never present to the eye of an observer a perfectly natural appearance, and hence this mode of representation is used only in drawings made for the development of the principles and the solution of problems in Descriptive Geometry, and for the purposes of mechanical or architectural constructions.

Whenever an accurate picture of an object is desired, the scenographic method must be used, and the position of the point of sight chosen as indicated in Art. 213.

274. Linear perspective. That application of the principles of Descriptive Geometry which has for its object the accurate representation, upon a single plane, of the details of the form and the principal lines of a body is called *Linear Perspective.*

Aerial perspective. The art by which a proper coloring is given to all parts of the representation is called aerial perspective. This, properly, forms no part of a mathematical treatise, and is therefore left entirely to the taste and skill of the artist.

275. General terms. The plane upon which the representation of the body is made is called the *plane of the picture;* and a point is represented upon it, as in all other cases (Art. 4), by drawing through the point and the point of sight a straight line. The

point in which it pierces the plane of the picture is the *perspective of the given point.*

These projecting lines are called *visual rays*, and when drawn to the points of any curve, form a *visual cone* (Art. 83).

Any plane passing through the point of sight is made up of visual rays, and is called a *visual plane.*

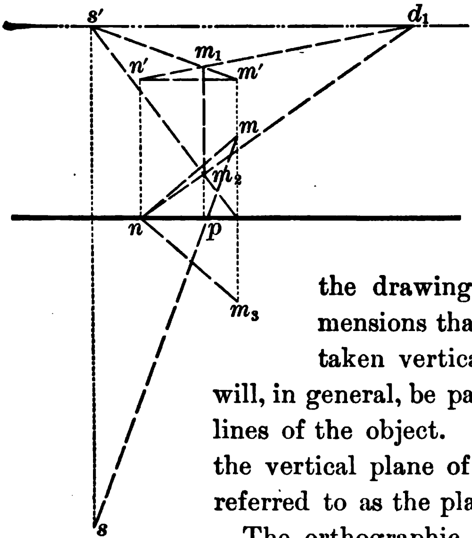


FIG. 123.

The plane of the picture is usually taken between the object to be represented and the point of sight, in order that the drawing may be of smaller dimensions than the object. It is also taken vertical, as in this position it will, in general, be parallel to many principal lines of the object. It may thus be used as the vertical plane of projection, and will be referred to as the plane V (Art. 17).

The orthographic projection of the point of sight, on the plane of the picture, is called the *principal point of the picture;* and a horizontal line through this point and in the plane of the picture is the *horizon of the picture.*

PERSPECTIVES OF POINTS AND STRAIGHT LINES. VANISHING POINTS OF STRAIGHT LINES

276. To find the perspective of a point. Let M, Fig. 123, be any point in space; let the ground line be the horizontal trace of the plane of the picture, and S the point of sight.

The visual ray SM, through M, pierces the plane of the picture V, in m_1 (Art. 24), which is the perspective of the point M.

277. To construct the perspective of a straight line. The indefinite perspective of a straight line, as MN, Fig. 124, may be drawn by finding the perspectives of any two of its points as in the preceding article, and joining them by a straight line. The point n', in which this line pierces V, is evidently its own perspective, and hence one point of the perspective of the line. If now a line be drawn through the point of sight to a point of the line MN at an infinite distance, it will be parallel to MN and its vertical piercing point, p', will be a second point in the perspective of the line, and $p'n'$ will be the perspective of the line.

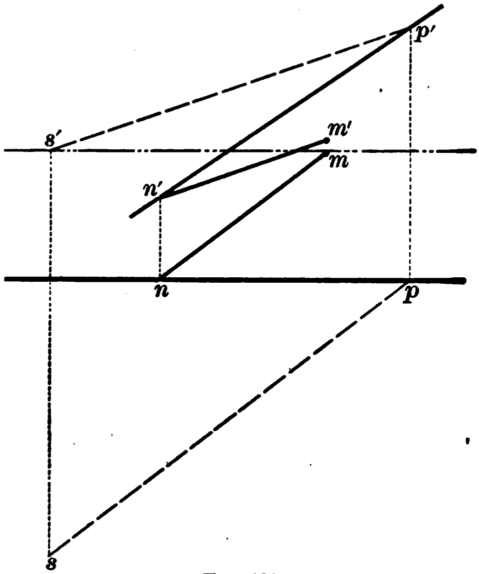


FIG. 124.

The perspective of that point of a line at an infinite distance is called the **vanishing point** of the line. Hence *to construct the perspective of a straight line, draw a straight line through the vanishing point and the point where the line pierces the plane of the picture.*

278. To find the vanishing point of a straight line, draw a line parallel to it through the point of sight, and find where this line pierces the plane of the picture.

The principal point is evidently the vanishing point of *all perpendiculars* to the plane of the picture, since a line through S parallel to any of these perpendiculars will pierce V in the principal point.

All horizontal lines vanish in the horizon. Why?

A system of parallel straight lines will have a *common vanishing point*, since the line through the point of sight parallel to one is parallel to all, and pierces V in a single point. Hence their perspectives all intersect at this point.

The vanishing point of all *lines parallel to the plane of the picture* is at infinity. Hence the perspective of a line parallel to the plane of the picture is parallel to the line itself. Let the student assume a line parallel to V and find its perspective.

What is true of the perspectives of all *vertical lines?*

If a straight line pass through the point of sight, it is a visual ray, and the *point* in which it pierces the plane of the picture is its perspective.

279. Vanishing line of a plane. If a plane be passed through the point of sight parallel to a given plane, it will contain all straight lines drawn through this point parallel to lines of the plane. Its trace on the plane of the picture will therefore contain the vanishing points of all these lines, and of lines parallel to them. This trace is the *vanishing line of the plane.*

The horizon is evidently the vanishing line of all horizontal planes. It therefore contains the vanishing points of all horizontal lines.

280. Diagonal. Any horizontal straight line making an angle of 45° with the plane of the picture is a diagonal.

Distance points. A straight line through S, Fig. 123, parallel to a diagonal, pierces V in the horizon, and at a distance from the principal point s' equal to the distance of S from the plane of the picture. Since two such lines can be drawn, one on each side of the perpendicular Ss', there will be two vanishing points of diagonals, one for those which incline to the right, and another for those which incline to the left.

These points are also called *points of distance*, since when

they are assumed or fixed, the distance of the point of sight from the plane of the picture is determined.

281. To find the perspective of a point when the horizontal projection of the point of sight lies outside the limits of the drawing. If a point be on a line, its perspective will be on the perspective of the line. If then through any point two straight lines be drawn, and if their perspectives be found as in Art. 277, the intersection of these perspectives will be the perspective of the given point. Hence, in practice, the perspective of a point is constructed *by drawing the perspectives of a diagonal and a perpendicular* which, in space, pass through the point, and *finding the intersection of these perspectives.*

Thus let s', Fig. 123, be the *principal point*, $s'd_1$ the horizon, d_1 one point of distance, and M any point in space.

MN is a diagonal through the point, its vertical projection $m'n'$ being parallel to the ground line (Prop. XIV, Art. 14). It pierces V at n', vanishes at d_1, and $n'd_1$ is its perspective (Art. 277). The perpendicular through M pierces V at m', vanishes at s', and $m's'$ is its perspective. These perspectives intersect at m_1, the perspective of M.

If the point M should be in the horizontal plane, the diagonal and perpendicular would pierce V in the ground line.

When the given point is near the horizontal plane through the point of sight, the perspectives of the diagonal and perpendicular through it are so nearly parallel that it is difficult to mark accurately their point of intersection. In this case, find the perspective of a vertical line through the given point; its intersection with the perspective of the diagonal or perpendicular will be the required point. Thus in Fig. 123, find m_2, the perspective of m, the foot of a vertical line through M, and draw m_2m_1 intersecting $m's'$ in m_1, the perspective of M.

282. Revolution of the horizontal projection. Since the object to be represented is usually behind the plane of the picture (Art.

275), and is given by its orthographic projections, these projections when made as in Art. 5 will occupy the same part of the drawing as the perspective, and cause confusion. To avoid this in some degree, that portion of the horizontal plane which is occupied by the horizontal projection of the object is revolved about the ground line 180°, until it comes in front of the plane of the picture. Thus, in Fig. 123, m comes to m_3, and the horizontal projection of the diagonal MN to the position m_3n, its vertical projection being in its primitive position, and the point n' being found as before (Art. 24). It will be observed that the horizontal projection of each diagonal, in this new position, lies in a direction contrary to that of its true position. The vanishing point will be determined by its true direction (Art. 278).

Perspectives of Curves

283. The perspective of any curve may be found by joining its points with the point of sight by visual rays, thus forming a visual cone. The intersection of this cone by the plane of the picture will be the required perspective. Or the perspectives of its points may be found as in Art. 281, and joined by a line.

If the curve be of single curvature (Art. 55), and its plane pass through the point of sight, its perspective will be a *straight line*.

284. If two lines are tangent, their perspectives will be tangent. For the visual cones, by which their perspectives are determined, will be tangent along the common rectilinear element passing through the point of contact of the lines; hence their intersection by the plane of the picture will be tangent at the perspective of this point of contact.

285. Perspective of a circle. If the circumference of a circle be parallel to the plane of the picture, its perspective will be the circumference of a circle whose center is the perspective of the given center; also, if it be so situated that the plane of the picture makes in its visual cone a subcontrary section (Art. 215).

In all other cases, when behind the plane of the picture, and when its plane does not pass through the point of sight, its perspective will be an *ellipse*.

It is evident that if the condition be not imposed that the circle shall be behind the plane of the picture, this plane may

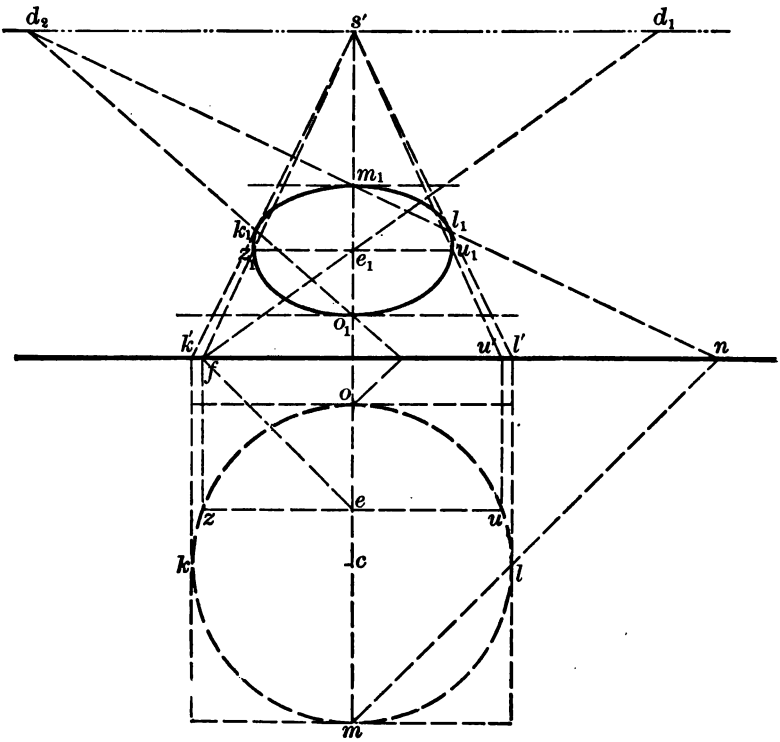


FIG. 125.

be so taken as to intersect the visual cone in *any of the conic sections*.

286. Perspective of a circle lying in a horizontal plane. Let *mlo*, Fig. 125, be a circle in the horizontal plane, revolved as in Art. 282, and let s' be the principal point, the point of sight

being taken in a plane through the center and perpendicular to the ground line, and let d_1 and d_2 be the *points of distance.*

o_1 is the perspective of o, and m_1 of m (Art. 281), and o_1m_1 of the diameter om. The perspectives of the two tangents at o and m will be tangent to the perspective of the circle at o_1 and m_1 (Art. 284); and since the tangents are parallel to the ground line, their perspectives will be parallel to the ground line (Art. 278), or perpendicular to o_1m_1; hence o_1m_1 is an *axis* of the ellipse. Through its middle point e_1 draw d_1e_1; it is the perspective of the diagonal fe, and e_1 is the perspective of e, and u_1z_1 of the parallel chord uz, $u's'$ and z_1s' being the perspectives

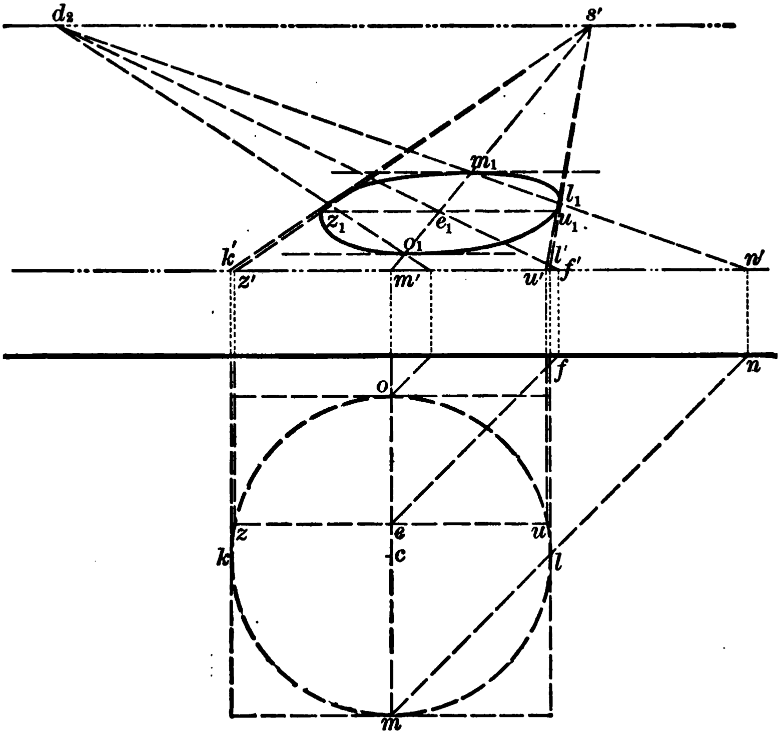


FIG. 126.

of the perpendiculars through u and z, and u_1z_1 is the other axis of the ellipse. On these two axes the ellipse can be described as in Art. 67.

The perspectives of the two perpendiculars ll' and kk' are tangent to the ellipse at l_1 and k_1.

If the point of sight be taken in any other position, the perspective of the circle, Fig. 126, may be determined by first finding the conjugate diameters, o_1m_1 and u_1z_1, and then constructing the ellipse as in Art. 257.

287. Line of apparent contour. If a body bounded by a curved surface be enveloped by a tangent visual cone, the line of contact of this cone will be the outer line of the body, as seen from the point of sight, and is the *line of apparent contour* of the body.

The perspective of this line (Art. 283) is evidently the bounding line of the picture or drawing, and is a principal line of the perspective.

When the body is bounded by plane surfaces, the visual cone will be made up of visual planes; the line of apparent contour will not be a line of mathematical contact, but will still be the outer line of the body as seen, and will be made up of straight lines.

When the body is irregular, or composed of broken surfaces, the line of contour may be composed partly of strict lines of contact, either straight or curved, and partly of lines not of contact, but still the outer lines of the body as seen. In all cases their perspectives will form the boundary of the picture.

To construct the perspective of any body, we must then determine the perspective of its line of apparent contour, and of such other principal lines as will aid in indicating its true form.

288. If a line of the surface intersect the line of contact of the visual cone, the perspective of this line will be tangent to the

perspective of the line of contact. For at the point of intersection draw a tangent to each of the two lines ; these tangents lie in a plane tangent to the surface, and this plane is also a visual plane tangent to the cone. Its trace on the plane of the picture is the perspective of both tangents, and tangent to the perspectives of both lines at a common point. They are therefore tangent to each other.

Vanishing Points of Rays of Light and of Projections of Rays

289. To find the vanishing point of rays of light. Since the rays of light are parallel, they will have a common vanishing point, which may always be determined by drawing a visual ray of light, and finding the point in which it pierces V (Art. 278). In the construction of a drawing or picture, the direction of the light is usually chosen by the draftsman or artist. This is done by assuming the

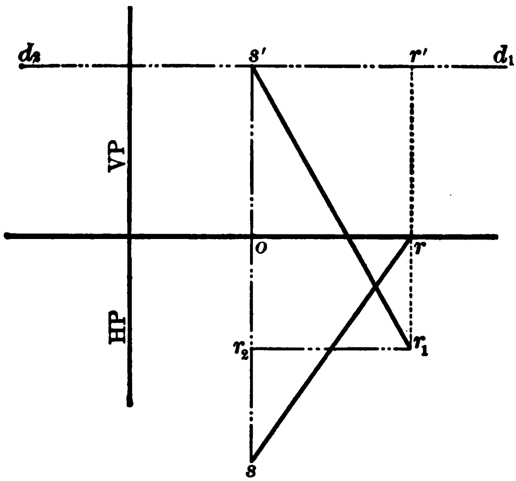


Fig. 127.

vanishing point of rays at once, as the direction of the light is thus completely determined. Thus let r_1, Fig. 127, be the vanishing point of rays, S being the point of sight; *sr* will be the horizontal projection of the ray, and $s'r_1$ its vertical projection.

290. To find the vanishing point of the horizontal projections of rays. The horizontal projections of rays of light being parallel and horizontal lines, must vanish in the horizon (Art. 278); and since a line through the point of sight parallel to the horizontal projection of a ray must be in the same vertical plane with a ray through the same point, it must pierce the plane of the picture in the vertical trace of this plane, that is, in the line r_1r' at r'. Hence, having assumed the vanishing point of rays, through it draw a straight line perpendicular to the horizon; this line will intersect the horizon in the vanishing point of the horizontal projections of rays.

291. To find the vanishing point of the profile projections of rays. The orthographic projections of rays on all planes perpendicular to the ground line, as the plane P, are parallel. The line $s's$ is the vanishing line of these planes (Art. 279), and must therefore contain the vanishing point of these projections. This point must also be in the trace of a plane of rays through S, perpendicular to these profile planes. r_1r_2, parallel to the ground line, is this trace; hence r_2 is the vanishing point of the profile projections of rays.

292. To find the vanishing point of the vertical projections of rays. The orthographic projections of rays on the plane of the picture or on planes parallel to it, are parallel; and being in or parallel to the plane of the picture, will be parallel in perspective (Art. 278), and all parallel to r_1s', the projection of the ray through S on V.

Perspectives of the Shadows of Points and Straight Lines on Planes

293. Since the *shadow of a point on a plane* is the point in which a ray of light through the point pierces the plane (Art. 250), it must be the intersection of the ray with its orthographic projection on the plane. Hence, **to construct the perspective of**

the shadow of a point on any plane, through the perspective of the point draw the perspective of a ray, and through the perspective of the projection of the point on the plane draw the perspective of the projection of the ray; the intersection of these two lines will be the perspective of the required shadow (Art. 281).

If the shadow be on the horizontal plane, join the perspective of the point with the vanishing point of rays r_1, Fig. 127, and the perspective of the horizontal projection of the point with the vanishing point of horizontal projections r'; the intersection of these two lines will be the perspective of the shadow of the given point.

If the shadow be on any profile plane, join the perspective of the projection of the point on this plane with the vanishing point of the projections of rays on profile planes r_2; the intersection of this line with the perspective of the ray will be the perspective of the shadow.

If the shadow be on any plane parallel to V, through the perspective of the projection of the point on this plane draw a line parallel to the projection of the ray on the plane of the picture $s'r_1$; its intersection with the perspective of the ray will be the perspective of the shadow.

If the shadow be on any vertical plane, draw through the perspective of the horizontal projection of the point a straight line to the vanishing point of horizontal projections. It will be the perspective of the horizontal trace of a vertical plane of rays through the point. At the point where it intersects the perspective of the horizontal trace of the given plane erect a vertical line; it will be the perspective of the intersection of the plane of rays with the given plane. The point where this intersects the perspective of the ray through the given point will be the perspective of the shadow.

294. The perspective of the indefinite shadow cast by a straight line on a plane may be constructed by finding the perspective

of the point in which the line pierces the plane, and joining it with the perspective of the shadow cast by any other point of the straight line; or by joining the perspectives of the shadows of any two points of the line by a straight line.

If the line be of definite length, join the perspectives of the shadows of its two extremities by a straight line.

Practical Problems

295. Problem 86. **To construct the perspective of an upright rectangular pillar, with its shade and shadow on the horizontal plane.**

Let *lmno*, Fig. 128, be the lower base of the pillar in the horizontal plane, revolved as in Art. 282, and $p'q'$ the vertical projection of the upper base, s' *the principal point*, d_2 one of *the points of distance*, r_1 the vanishing point of rays, and r' of horizontal projections of rays. These important points will be thus represented in all the following problems.

Since ml and no are perpendicular to V, they vanish at s' (Art. 278), and $m's'$ and $n's'$ are their indefinite perspectives; id_2 is the perspective of the diagonal ni, and n_1 of the point n (Art. 281). n_1m_1, parallel to the ground line, is the perspective of mn (Art. 278); o_1 of the point o; n_1o_1 of the edge no (Art. 277); o_1l_1 of ol; and m_1l_1 of ml.

The edges of the upper base, which are horizontally projected in on and lm, pierce V at p' and q', and vanish at s'; hence $p's'$ and $q's'$ are their indefinite perspectives. The diagonal of the upper base, horizontally projected in ni, pierces V at i', $i'd_2$ is its perspective, and p_1 the perspective of the vertex of the upper base horizontally projected in n, and v_1 of that horizontally projected at l. The vertical edges which pierce H in m, n, o, and l, are parallel to V, and their perspectives parallel to themselves; hence, m_1q_1, n_1p_1, o_1u_1, l_1v_1 are

their perspectives terminating in the points q_1, p_1, u_1, v_1, and $q_1p_1u_1v_1$ is the perspective of the upper base.

The face of which $n_1o_1u_1p_1$ is the perspective is in the shade, and therefore is darkened in the drawing.

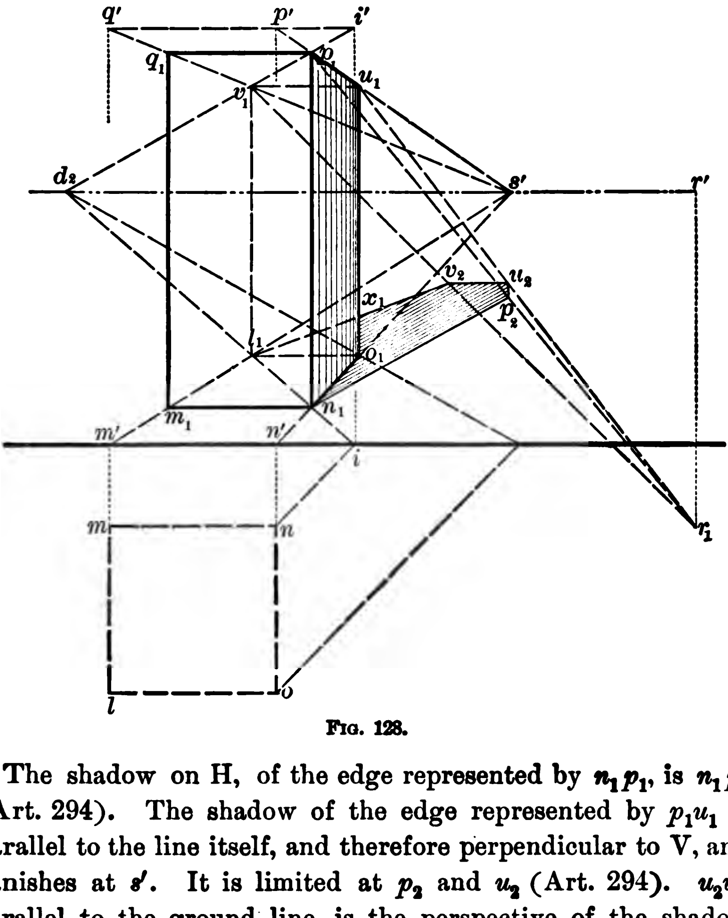


FIG. 128.

The shadow on H, of the edge represented by n_1p_1, is n_1p_2 (Art. 294). The shadow of the edge represented by p_1u_1 is parallel to the line itself, and therefore perpendicular to V, and vanishes at s'. It is limited at p_2 and u_2 (Art. 294). u_2v_2, parallel to the ground line, is the perspective of the shadow of the edge represented by u_1v_1, and l_1v_2 of that represented by l_1v_1.

That part of the drawing within the line $n_1p_2v_2x_1$ is darkened, being the perspective of that part of the shadow on **H** which is seen.

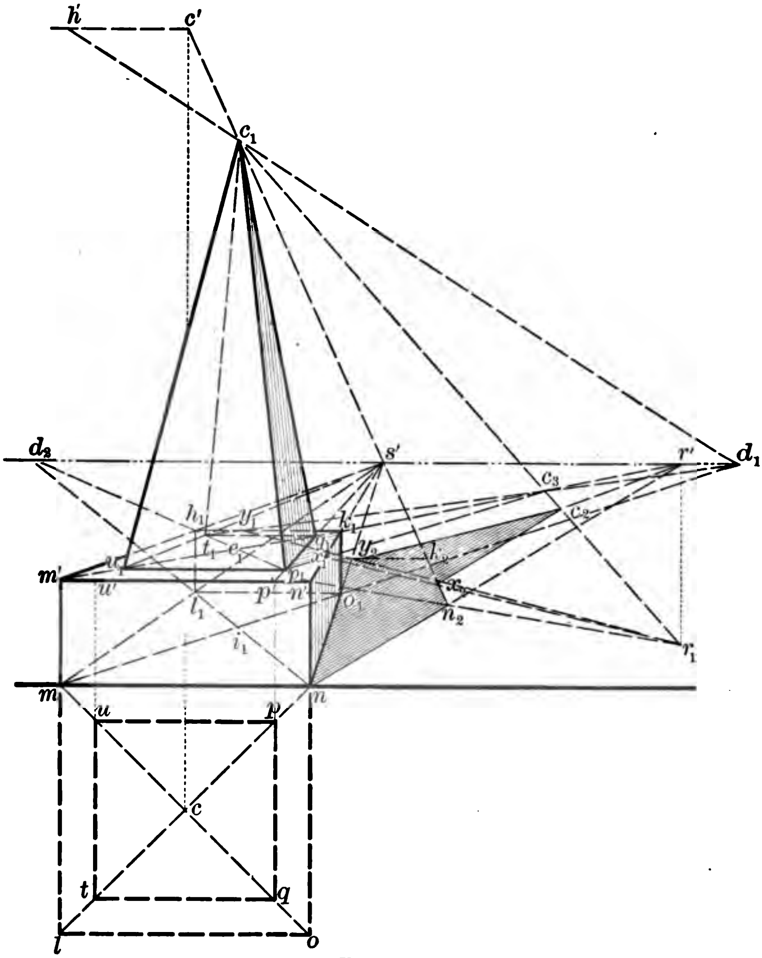


FIG. 129.

296. PROBLEM 87. **To construct the perspective of a square pyramid with its pedestal, and the perspective of its shadow.**

Let *mnol*, Fig. 129, be the base of the pedestal revolved, as in Art. 282, and $mnn'm'$ its front face in the plane of the picture, *pqtu* the horizontal, and $p'u'$ the vertical projection of the base of the pyramid, and C its vertex.

The face $mnn'm'$, being in the plane of the picture, is its own perspective. The four edges of the pedestal, which are perpendicular to V, pierce it in m, n, n', and m', and vanish at s'. The two diagonals (mo, mn) and $(mo, m'n')$ pierce V at m and $m,'$ and vanish at d_1. The diagonals (nl, nm) and $(nl, n'm')$ pierce V at n and n', and vanish at d_2. Hence, o_1 is the perspective of (o, n) (Art. 281); k_1 of (o, n'); l_1 of (l, m); and h_1 of (l, m'); and the perspective of the pedestal is drawn as in Art. 295.

The two perpendiculars (ut, u') and (pq, p') pierce V at u' and p'; $u's'$ and $p's'$ are their perspectives intersecting $m'd_1$ and $n'd_2$ in u_1, q_1, p_1, and t_1, and $u_1p_1q_1t_1$ is the perspective of the base of the pyramid.

The perpendicular through C pierces V at c', and a diagonal through the same point at h', and $c's'$ and $h'd_1$ are their perspectives, and c_1 the perspective of C; and c_1u_1, c_1p_1, c_1q_1, and c_1t_1 are the perspectives of the edges of the pyramid.

nn_2 is the perspective of the shadow of nn' on H; n_2k_2 of the shadow of the edge represented by $n'k^1$ (Art. 278). i_1 is the perspective of c, the horizontal projection of the vertex, i_1r' (not shown in figure) of the horizontal projection of a ray through C, and c_1r_1 the perspective of the ray; hence, c_2 is the perspective of the shadow of C on H (Art. 293). e_1 is the perspective of the projection of C on the upper base of the pedestal, e_1r' of the projection of a ray on this plane, and c_3 of the shadow cast by C on this plane, and p_1c_3 the perspective of the shadow cast by the edge CP on this plane (Art. 294). This shadow passes from the upper base at a point of which x_1 is the perspective; x_1r_1 is the perspective of a ray through this point,

intersecting n_2s' at x_2, the perspective of the shadow cast by one point of the edge CP on H, and x_2c_2 is the perspective of the shadow.

t_1c_3 is the perspective of the shadow cast by the edge CT on the upper base of the pedestal, y_1 of the point at which it passes from this base, y_2 of the shadow of this point on H, and y_2c_2 of the shadow of the edge CT.

The face represented by $c_1p_1q_1$ is in the shade, as also that represented by $nn'k_1o_1$, and both are darkened on the drawing.

$p_1x_1k_1y_1t_1$ bounds the darkened part on the perspective of the upper base, and nn_2 - - - c_2y_2 - - - l_1 that on H.

297. PROBLEM 88. **To construct the perspective of a cylindrical column with its square pedestal and abacus, and also the shade of the column and shadow of the abacus on the column.**

Let $mngl$, Fig. 130, be the horizontal projection of both pedestal and abacus (Art. 282), $mm'n'n$ the vertical projection of the pedestal, and $e'l'g'f'$ that of the abacus, $upqt$ being the horizontal projection of the column, $m'n'$ the vertical projection of its lower, and $e'f'$ of its upper, base, the plane of the picture being coincident with that of the front faces of the pedestal and abacus.

Let the point of sight be taken as in Art. 286, in a plane through the centers of the upper and lower bases of the column and perpendicular to the ground line.

Construct the perspective of the pedestal as in Art. 295, $mm'n'n$ being its own perspective, and $m'l_1$ and $n'g_1$ the perspectives of those edges of the upper face of the pedestal which pierce V at m' and n'.

In the same way construct the perspective of the abacus, $e'l'g'f'$ being its own perspective, and $e's'$ and $f's'$ the indefinite perspectives of the edges of the lower face which pierce V at e' and f'.

$u_1t_1q_1p_1$ is the perspective of the lower base of the column determined as in Art. 286, u_1q_1 being its conjugate and t_1p_1

its transverse axis. In the same way the perspective of the upper base is determined, its conjugate axis being the per-

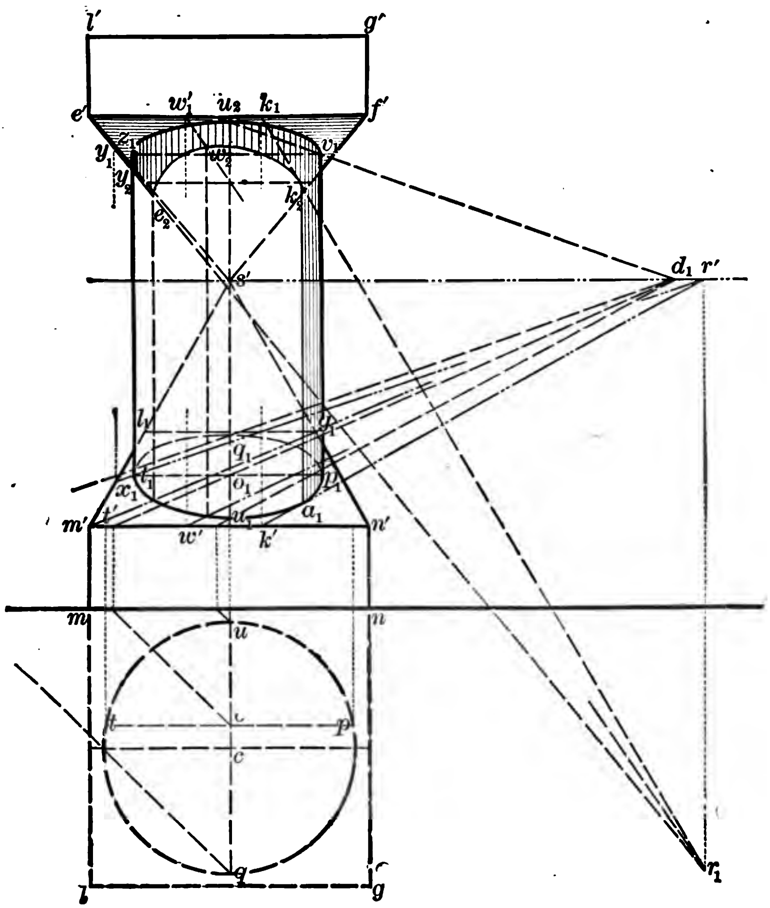


FIG. 130.

spective of that diameter which pierces V at u_2, horizontally projected in uq, and its transverse axis z_1v_1 the perspective of the chord corresponding to tp.

t_1z_1 and p_1v_1 tangent to these ellipses (Art. 288), and per-

pendicular to the ground line, are the perspectives of the elements of contour of the column, and complete its perspective (Art. 287).

The elements of shade on the column are determined by two tangent planes of rays (Art. 262), and since these planes are vertical, their intersections with the plane of the upper face of the pedestal will be parallel to the horizontal projections of rays, and therefore vanish at r'. Since these intersections are also tangent to the lower circle of the column, their perspectives will be tangent to the ellipse $t_1u_1p_1$. Hence, if through r' two tangents be drawn to $t_1u_1p_1$, their points of contact will be points of the perspectives of the elements of shade. a_1k_2 is the perspective of the only one which is seen. The plane of rays by which this is determined intersects the lower edge $e'f'$ of the abacus in k_1, through which draw k_1r_1. It intersects a_1k_2 in k_2, the perspective of the point of shadow cast by k_1, and limiting the perspective of the element of shade.

Draw $r't_1$; it is the perspective of the intersection of a vertical plane of rays with the upper face of the pedestal. This plane intersects the column in an element represented by t_1z_1, and the edge represented by $e's'$ in a point of which y_1 is the perspective. Through y_1 draw y_1r_1 intersecting t_1z_1 in y_2, the perspective of the shadow cast on the column by the point Y, and in the same way the perspective of the shadow cast by any point of the same edge may be determined.

Through m' draw $m'r'$, and through e' draw $e'r_1$, intersecting the perspective of the element cut from the column in e_2, the perspective of the shadow cast by e' on the column, and in the same way the perspective of the shadow cast by any other point of the edge $e'f'$ may be found, as w_2 the perspective of the shadow cast by w'_1.

All of the lower face of the abacus is in the shade and is darkened in the drawing. The line y_2e_2 - - - k_2 is the perspective

of the line of shadow on the column, and all above it is darkened, as all beyond a_1k_2. a_1r' is the perspective of the shadow cast by the element of shade on the upper surface of the pedestal, and the part beyond it which is seen is therefore darkened.

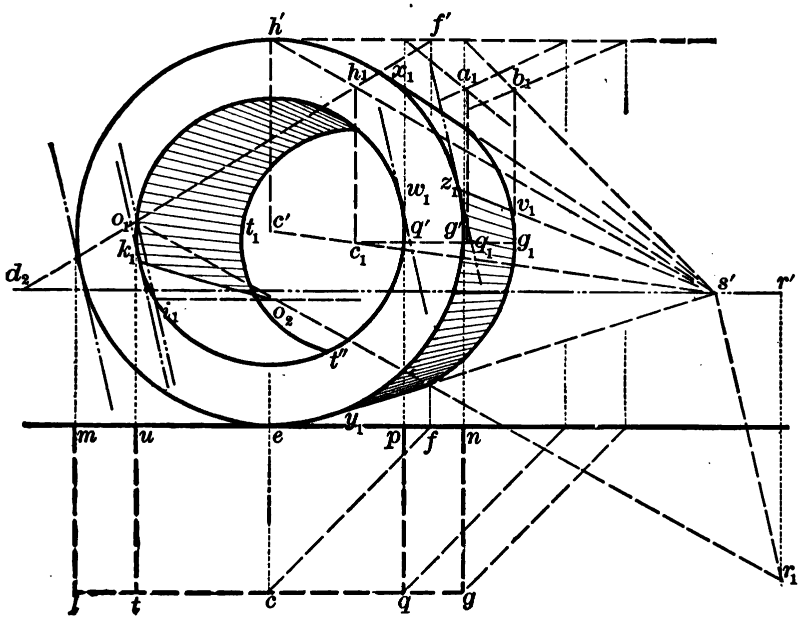


FIG. 131.

298. PROBLEM 89. **To construct the perspective of an upright cylindrical ring with its shade and shadow on its interior surface.**

Let *mngl*, Fig. 131, be the horizontal projection of the outer cylinder of the ring, and ez_1h' its vertical projection; *upqt* the horizontal, and $o_1i_1w_1$ the vertical projection of the inner cylinder, the plane of the picture being coincident with the plane of the front face of the ring.

The two circles ez_1h' and $o_1i_1w_1$, being in the plane of the picture, are their own perspectives.

To construct the perspectives of the back circles horizontally projected in lg and tq, draw the vertical radius (c, $c'h'$) and the two vertical tangents at Q and G. h_1c_1, a_1q_1, and b_1g_1 are the perspectives of these lines (Art. 281), and c_1, q_1, and g_1 are the perspectives of the points C, Q, and G. With c_1 as a center, and c_1q_1 and c_1g_1 as radii, describe the arcs t_1t'' and v_1g_1; they will be the perspectives of arcs of the back circles.

$s'y_1$ and $s'x_1$ are the perspectives of the elements of contour tangent to the circles ey_1x_1 and v_1g_1 (Art. 288).

The *element of shade* on the outer cylinder is determined by a tangent plane of rays. This plane being perpendicular to V, its vertical trace will be parallel to r_1s' (Art. 292), and tangent to y_1z_1h' at z_1. z_1v_1 will be the perspective of this element.

Points of the shadow cast by the circle $o_1i_1w_1$ on the interior cylinder will be found by passing planes of rays perpendicular to the plane of the picture. Each plane will intersect the circle in a point which casts a shadow on the element which the plane cuts from the cylinder (Art. 265). o_1i_1 parallel to r_1s' is the vertical trace of such a plane, intersecting the circle in o_1, and the cylinder in an element of which i_1s' is the perspective. o_1r_1 is the perspective of a ray through o_1, and o_2 is the perspective of the shadow. The perspective of the shadow evidently begins at k_1.

299. PROBLEM 90. **To construct the perspective of an inverted frustum of a cone with its shade and shadow.**

Let C, Fig. 132, be the vertex, $kolm$ the horizontal (Art. 282), and $k'l'$ the vertical projection of the upper base, $ehgf$ the horizontal, and $e'g'$ the vertical projection of the lower base, and $k'e'$ the vertical projection of one of the extreme elements.

The perspectives of the bases are determined as in Art. 286, $k_1o'y_1m_1$ of the upper, and $e_1h_1a_1f_1$ that of the lower base.

The perspective of the vertex is found as in Art. 281, CZ being the diagonal through it, piercing V at z'; and (co, c') the

perpendicular piercing V at c'; $z'd_2$ and $c's'$ are their perspectives intersecting at c_1. Through c_1 draw the two tangents c_1w_1 and c_1y_1; they are the perspectives of the elements of contour, and

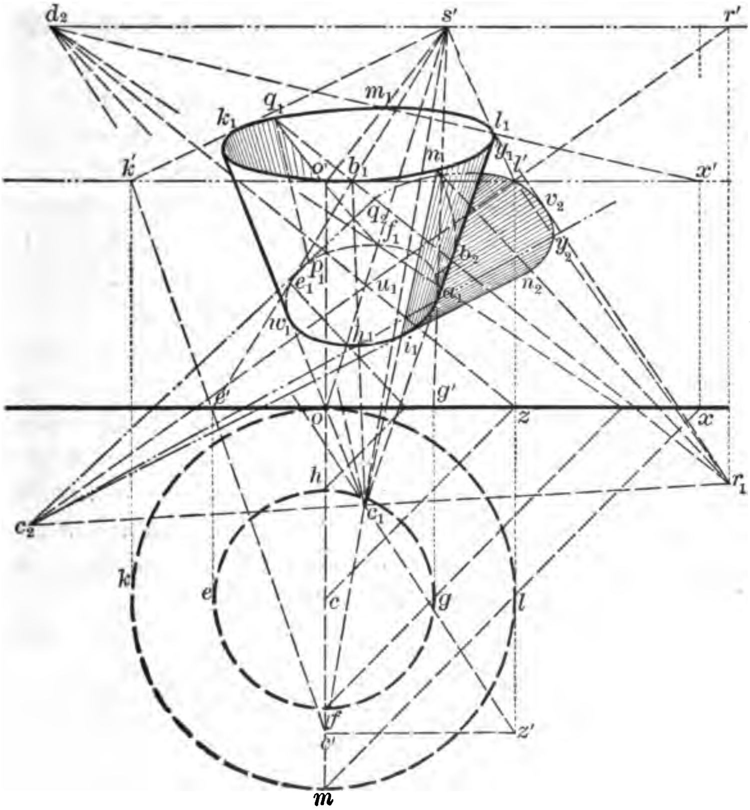


FIG. 132.

with the curves $k_1o'y_1m_1$ and $e_1h_1a_1f_1$ limit the perspective of the frustum.

The *elements of shade* on the cone are determined by two tangent planes of rays intersecting in a ray of light passing through the vertex (Art. 101); r_1c_1 is the perspective of this

ray, and u_1r' the perspective of its horizontal projection, u_1 being the perspective of c; hence c_2 is the perspective of the point in which this ray pierces H, and the tangents c_2i_1 and c_2p_1 are the perspectives of the horizontal traces of the tangent planes, and i_1n_1 and p_1q_1 of the elements of shade (Art. 263), the former only being seen.

That part of the *circumference of the upper base* toward the source of light, and between the points of which n_1 and q_1 are the perspectives, casts its shadow on the interior of the cone. Points of this shadow may be determined by intersecting the cone by planes of rays through the vertex. Each plane intersects the circumference in a point, and that part of the cone opposite the source of light in a rectilinear element. A ray of light through the point intersects the element in a point of the required shadow.

c_2a_1 is the perspective of the horizontal trace of such a plane. The plane intersects the cone in two elements, represented by a_1y_1 and b_1c_1, and b_1 is the perspective of the point in which the plane intersects the circumference, b_1r_1 the perspective of a ray through this point, and b_2 the perspective of the required point of shadow.

The curve of shadow evidently begins at the points of which n_1 and q_1 are the perspectives.

The perspective of the lowest point of this shadow will be found by using the line c_2u_1, as this is the perspective of the trace of that plane which cuts out the element farthest from the point casting the shadow.

The perspective of the point of shadow on any element is found by using the line drawn through c_2 and the lower extremity of the perspective of the element. Thus the point of tangency, b_2 (Art. 288), is found by using c_2a_1 as the perspective of the trace of the auxiliary plane.

q_1k_1o' is the perspective of that part of the shadow on the

interior which is seen, and is therefore darkened in the drawing, as is the perspective of the shade $n_1i_1a_1y_1$.

i_1n_2 and p_1q_2 are the perspectives of the shadows cast by the elements of shade on H, the points n_2 and q_2 being determined by n_1r_1 and q_1r_1.

The plane determined by c_2a_1 intersects the circumference of the upper base in a point of which y_1 is the perspective. y_1r_1 is the perspective of a ray through this point, piercing H at a point of which y_2 is the perspective. This is a point of the perspective of the curve of shadow of the upper circumference on H; and in the same way any number of points may be determined.

r_1v_2 drawn tangent to $n_1y_1m_1$ will also be tangent to $n_2y_2q_2$, since this tangent is the perspective of the element of contour of the cylinder of rays through the upper circumference, by which its shadow is determined (Art. 288).

The curve $n_2y_2q_2$ is also tangent to i_1n_2 and p_1q_2 at n_2 and q_2.

300. PROBLEM 91. **To construct the perspective of a niche with its shadow on its interior surface.**

Let the niche be formed by a semicylinder, the lower base of which is mlo, Fig. 133, and the upper base vertically projected in $m'o'$, and the quarter sphere vertically projected in the semicircle $m'k'o'$, its lower semicircle being coincident with the upper base of the cylinder, and the plane of the picture so taken as to contain the elements mm' and oo' and the front circle $m'k'o'$.

These elements and front circle, being in the plane of the picture, will be their own perspectives.

An arc of an ellipse, ml_1o, is the perspective of the lower base of the cylindrical part, and $m'n_1o'$ that of the upper base, these being constructed as in Art. 286, l_1 being the perspective of l, and n_1 of the corresponding point of the upper base. These lines form the perspective of the niche.

The lines which cast shadows on the interior of the niche are the element mm' and the arc $m'k'e'$.

The shadow cast by mm′ is determined by passing through it a vertical plane of rays; mr is the perspective of its trace

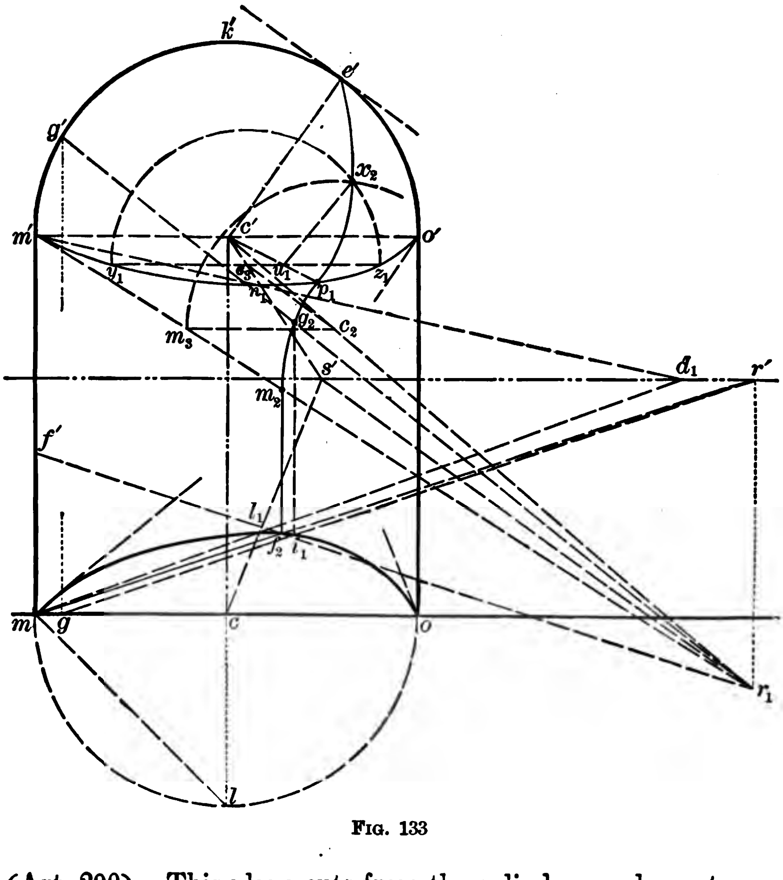


FIG. 133

(Art. 290). This plane cuts from the cylinder an element represented by f_2m_2, the intersection of which by $m'r_1$ in m_2 is the perspective of the shadow cast by m' on the element; and f_2m_2 is the perspective of the shadow cast by $m'f'$, a part of mm', on the

interior of the cylinder; mf_2 is the perspective of the shadow cast by mf' on H.

Points of the shadow cast by the front circumference, $m'k'e'$, on the cylindrical part of the niche may be determined by intersecting the cylinder by vertical planes of rays. Each plane intersects the cylinder in a rectilinear element and the arc in a point. A ray of light through this point intersects the element in a point of the curve of shadow. $r'g$ is the perspective of the horizontal trace of such a plane. It intersects $m'k'e'$ in g', and the cylinder in an element represented by i_1g_2, and g_2 is the perspective of the shadow cast by g' on the element.

The shadow cast by a part of the front circumference on the spherical part of the niche is an equal arc of a great circle. For this shadow is determined by a cylinder of rays through the front circumference, and the part of each element of the cylinder between the point casting the shadow and its shadow is a chord of the sphere, and all these chords will be bisected by a plane through c' perpendicular to them. These half chords may be regarded, one set as the ordinates of the arc casting the shadow, and the other as the ordinates of the shadow, and since the corresponding ordinates of the two arcs are equal, and in all respects alike situated, the two arcs will be equal. Their planes evidently intersect in the radius $c'e'$ perpendicular to the axis of the cylinder of rays, or to the ray of light, and also perpendicular to the projection $s'r_1$ of the ray of light on the plane of the picture (Art. 289).

Points of this shadow may be constructed by intersecting the quarter sphere by planes parallel to the plane of the picture. Each plane will cut from the quarter sphere a semicircumference. The front semicircumference will cast upon this plane a shadow parallel and equal to itself (Art. 256), and the intersection of this with the semicircumference cut from the sphere will be a point of the required shadow (Art. 253).

Draw y_1z_1 parallel to $m'o'$; it may be taken as the perspective of the intersection of an auxiliary plane with the upper base of the cylinder, and the semicircumference $y_1x_2z_1$ is the perspective of the semicircumference cut from the sphere.

The center of the front circle casts a shadow upon this plane whose perspective is c_2 (Art. 293), c_3 being the perspective of the projection of c' on this plane, and c_3c_2 parallel to $s'r_1$ the perspective of the projection of the ray through c' on this plane. c_2m_3 parallel to $c'm'$ is the perspective of the shadow cast by the radius $c'm'$ on this plane (Art. 278), and m_3x_2 the perspective of the arc of shadow intersecting $y_1x_2z_1$ in x_2, a point of the perspective of the shadow.

The point at which the curve of shadow passes from the spherical to the cylindrical part is evidently the point in which the intersection of the plane of the curve of shadow with the plane of the upper base meets the circumference of the upper base. To determine the perspective of this point, through x_2 draw x_2u_1 parallel to $c'e'$; it is the perspective of a line of the plane of the curve of shadow as also of the plane of the circle of which m_3x_2 is the perspective. u_1 is the perspective of the point where this line pierces the plane of the upper base, and $c'u_1$ the perspective of the intersection of the plane of the circle of shadow with the plane of the upper base, and p_1 the perspective of the required point.

301. PROBLEM 92. **To construct the perspective of a circle lying in a given oblique plane, its diameter and its center being known.**

Let T, Fig. 134, be the plane in which the circle lies, and oo' (assumed as in Art. 28) the center of the circle. Let the plane be revolved into V about its vertical trace. The horizontal trace will take the position Tg_3f_3 (Art. 34), the center of the circle will fall at o_3, and the circle $a_3d_3b_3c_3$ may then be drawn.

Analysis. If two tangents be drawn to the circle in its revolved position, parallel to the V trace of the plane, they will

be parallel to the picture plane; their perspectives will be parallel to each other (Art. 278), and tangent to the perspective of the circle (Art. 284). The line joining the perspectives of the points of tangency may be taken as one diameter of the ellipse (Art. 285), its conjugate being drawn through its middle point and parallel to the tangents already drawn (Art. 257). This conjugate is the perspective of some chord of the circle parallel to the V trace of the plane. If this chord be found, and then the perspective of its extremities determined, we shall have the extremities of the second conjugate diameter, and the ellipse may be constructed.

Construction. Draw the diameter a_8b_8 perpendicular to VT. Its perspective will be one of the conjugate diameters of the required ellipse. The perspective of one of its extremities, a_8, is found by passing through it two lines, one parallel to the V trace, and one parallel to the H trace (in its revolved position), and finding the intersection of their perspectives.

Draw a_8f_8 parallel to VT. It cuts the H trace in a point whose revolved position is f_8, and whose original position is at ff'. The perspective of this point will lie in Tx, the perspective of the H trace (Arts. 277, 278). It will also lie in $f's'$, the perspective of a perpendicular to the picture plane through F (Art. 281). It must, therefore, be at f_1. Then f_1i, parallel to VT, is the indefinite perspective of a_8f_8 (Art. 278).

Now draw a_8e_1 parallel to the H trace in its revolved position. It cuts VT in e_1, which is its own perspective. Since this line, in its original position, is parallel to the H trace, it will vanish at x. Its perspective is, therefore, e_1x, which cuts f_1i in a_1. The perspective of a_8q is evidently a_1q, since q is its own perspective. It contains the perspective of b_8, found by drawing b_8n_1 parallel to Tf_8, and n_1x its perspective. Through the middle point, m_1, of a_1b_1 draw xm_1k_1 and then k_1m_8 parallel to Tf_8. m_8 is then the point whose perspective is m_1, and c_8d_8

is the chord whose perspective is the other conjugate diameter c_1d_1. These points c_1 and d_1 are found by drawing d_3j_1 and c_3p_1 parallel to Tf_3 and then drawing their perspectives j_1x and p_1x. These lines determine the extremities of the diameter c_1d_1 conjugate to a_1b_1. The ellipse may now be constructed upon these two diameters.

302. To construct the perspective of any curve lying in a given oblique plane, without reference to its orthographic projections. The method used for finding the perspective of A may be applied to any number of points, either on a circle or on *any other plane curve.*

303. Special cases. The above process for constructing the perspective of a circle will be somewhat simpler in the case of a circle parallel to the horizontal, the vertical, or the profile plane; but in *all* cases the following principle will hold: **The first conjugate diameter is always the perspective of that diameter of the circle which is perpendicular to the vertical trace of the plane in which the circle lies.** The second bisects the first, and is the perspective of a chord of the circle *parallel* to the V trace.

If the circle is parallel to the picture plane, its perspective will be a circle, and it is only necessary to find the perspective of its center and of one point on its circumference. The perspective can then be described with the compasses.

304. PROBLEM 93. **To construct the perspective of an oblique cylindrical ring with its shades and shadows.**

Let the lower circle of the ring rest in the plane T, **Fig. 134,** its perspective $a_1d_1b_1c_1$ having been found as in Art. 301.

To construct the perspective of the upper circle of the ring, erect a perpendicular to the plane T at any point in the diameter AB, as oo', and lay off a distance OR equal to the required distance between the planes of the two circles. The perspective of O is o_1, the intersection of $o's'$ with a_1b_1. The perpendicular vanishes at l', **the vanishing point of elements (Art. 278),**

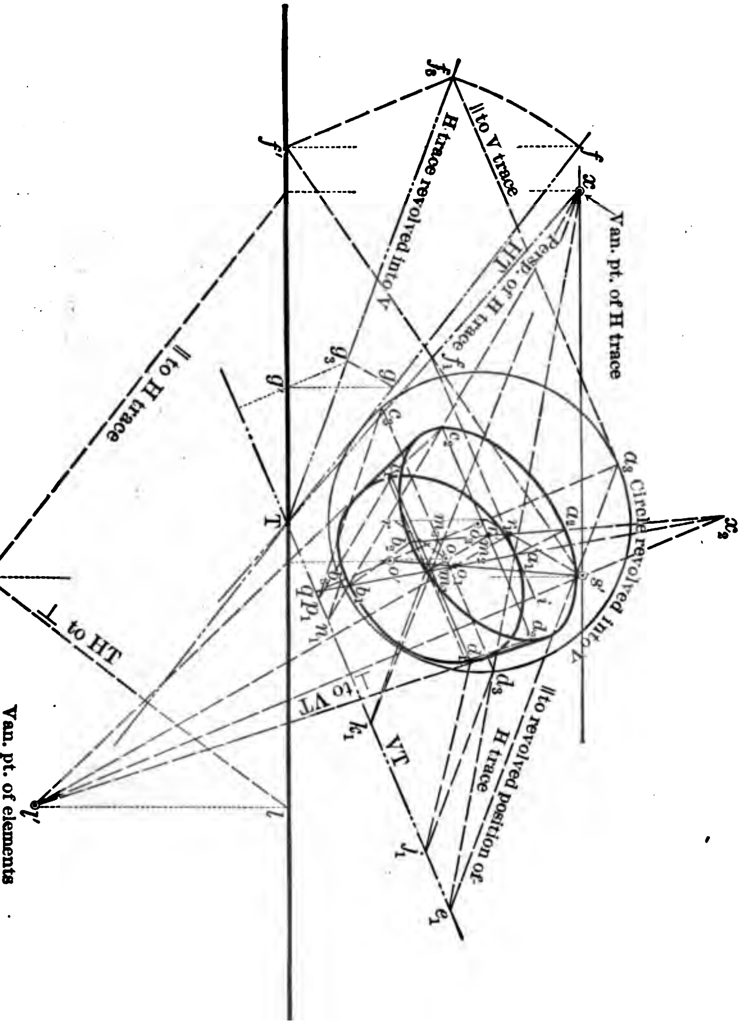

Van. pt. of H trace
|| to V trace
H trace revolved into V
Persp. of H trace
Circle revolved into V
|| to revolved position of H trace
|| to H trace
⊥ to HT
⊥ to VT
VT
Van. pt. of elements

FIG. 134.

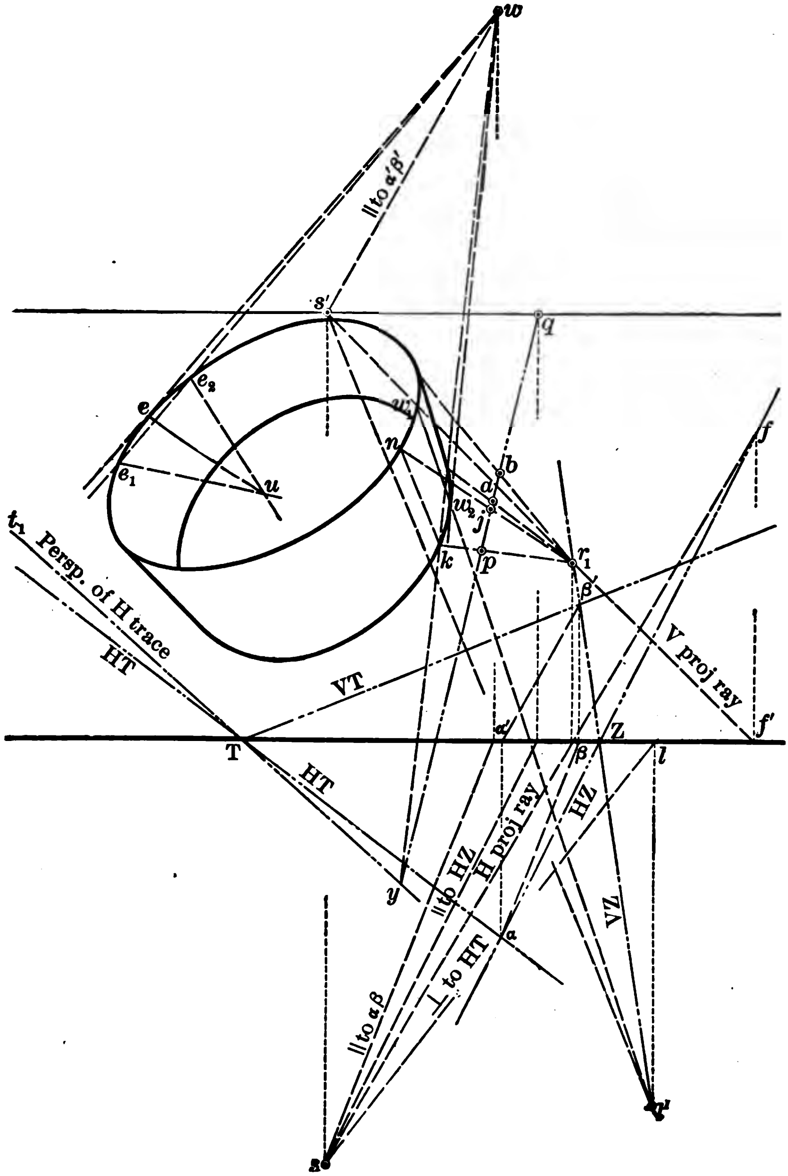
and the perspective of R is r_1 which is a point of the perspective of the diameter of the upper circle parallel to AB of the lower circle. These diameters vanish at x_2, found by drawing $s'x_2$ parallel to the V projection of the diameters, and producing b_1a_1, the perspective of the lower diameter, to meet it. The perspective of the upper diameter then lies along r_1x_2, its extremities a_2 and b_2 being found by producing the elements $l'a_1$ and $l'b_1$ to meet r_1x_2.

The other conjugate diameter of the ellipse is drawn parallel to VT through the middle point (m_2) of a_2b_2. The perspective of its projection on the plane T will be parallel to VT and pass through a point (near m_1) where m_2l' crosses a_1b_1. If through the points where this line intersects the perspective of the lower circle, lines be drawn through l' and extended to meet the line through m_2 parallel to VT, the intersections c_2 and d_2 will be the extremities of the conjugate diameter. The ellipse may now be constructed upon these diameters, and the elements of apparent contour drawn tangent to the two ellipses and vanishing at l'.

To find the shades and shadows of the ring, pass planes of rays parallel to the elements of the cylinder. Each will cut a line from H vanishing at q, Fig. 135, lines from the upper and lower bases vanishing at w, elements from the cylinder vanishing at l', and rays of light vanishing at r_1.

Through any point in space, as S, draw a ray of light SF, and a line SL parallel to the elements. Pass a plane Z through these two lines. It is parallel to all planes of rays cutting elements from the ring. The H traces of these planes vanish at q. The intersection between these planes and the planes of the circles will be parallel to ($a\beta$, $a'\beta'$) and vanish at w. Rays of light vanish at r_1.

The element of shade, w_1w_2, is found by passing a plane of rays tangent to the cylinder and finding the element of contact.


w
|| to α'β'
s'
q
e
e₂
e₁
u
w₁
n
b
d
w₂
j
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p
r₁
f
t₁ Persp. of H trace
HT
β'
V proj ray
VT
T
HT
d'
Z
β
l
f'
HZ
H proj ray
|| to HZ
y
a
⊥ to HT
VZ
|| to aβ
s
v'

FIG. 135.

This plane cuts the plane of the upper base in a line whose perspective is ww_1, tangent to the circle at w_1.

To construct the perspective of the shadow cast by the upper circle upon the interior of the ring, draw we tangent to the perspective of the upper circle at e. This is the point from which the perspective of the shadow starts. A plane of rays cutting the upper plane in the line whose perspective is we_2e_1, and the cylinder in the element whose perspective is e_2l', will contain a ray of light whose perspective is e_1r_1 and which intersects e_2l' in u, a point of the perspective of the required shadow.

The perspective of the shadow cast upon H is found as follows: A plane of rays cutting the lower plane of the ring in the line whose perspective is wky, the cylinder in the element whose perspective is kn, and the H plane in the line whose perspective is yq, will contain rays of light whose perspectives are kr_1 and nr_1, which pierce H in the points whose perspectives are p and j, respectively. These are the perspectives of the shadows of k and n. Similarly for the shadows of other points of the ring.

305. PROBLEM 94. **To construct the perspective of a sphere, its line of shade, and shadow on H.**

Join S, the point of sight, with C, the center of the sphere, and revolve the horizontal projecting plane of this line about the horizontal projecting line of C until it is parallel to V. This plane will cut from the sphere a great circle, and from the visual cone two elements tangent to the circle at points which, if joined, will give the diameter of the *circle of apparent contour;* its center O being, in its revolved position, at the intersection of this diameter and the line CS. Revolve CS and the point O to their original positions and pass a plane T through O perpendicular to CS (Art. 44). It is the plane of the circle of apparent contour. The perspective of this circle is the per-

spective of the sphere, and may be found by the method of conjugate diameters (Art. 301).

Pass a plane U through C and perpendicular to the rays of light (Art. 44). It cuts from the sphere a great circle which is the *line of shade.* Its perspective is found by the method of conjugate diameters (Art. 301).

The shadow of this circle upon H, found as in Problem 93, is the ***shadow of the sphere.***

When the V projection of the point of sight is coincident with c', the solution will be somewhat simplified, as the perspective of the sphere will then be a circle.

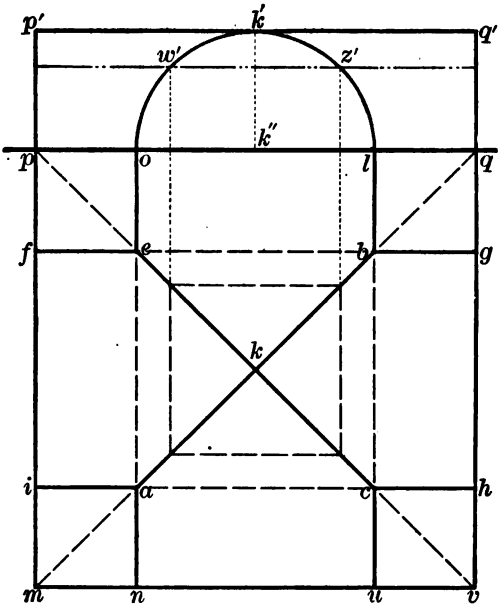

FIG. 136.

306. PROBLEM 95. **To construct the perspective of a groined arch with its shadows.**

To form the groined arch here considered, two equal right semicylinders, with semicircular bases, are so placed that their axes shall be at right angles and bisect each other and the diameters of the semicircular bases in the same horizontal plane. Thus, Fig. 136, let *nolu* be the horizontal, and *ok′l* the vertical, projection of one semicylinder, and *fghi* the horizontal, and *pp′q′q* the vertical, projection of the second cylinder, *pp′* and *qq′* being the vertical projections of the two semicircular bases.

These two cylinders intersect in two equal ellipses horizontally projected in the diagonals *ab* and *ce*. These ellipses are the *groins*. The arch is now formed by taking out from each cylinder that part of the other which lies within it. Thus, all that part of the cylinder whose elements are parallel to V, horizontally projected in *akc* and *bke*, is removed, as also that part of the other cylinder projected in *ake* and *bkc*.

The arch thus formed is placed upon four pillars standing in the four corners of a square, whose horizontal projections are the four squares *mnai*, *uchv*, etc., the elements *ih* and *fg* being coincident with the inner upper edges of the pillars which are horizontally projected in *ia-ch* and *fe-bg*, and the elements *no* and *ul* coincident with the inner upper edges which are projected in *na-eo* and *uc-bl*.

In this position of the arch the front circumference, or front base of the cylinder whose elements are perpendicular to V, springs from the upper extremities of the two edges of the front pillars which are horizontally projected at *n* and *u*, and is described on a diameter equal and parallel to *nu*. The back circumference, or other base of the same cylinder, springs from the upper extremities of the edges horizontally projected at *o* and *l*.

The side circumferences, or bases of the other cylinder, spring, the one from the upper ends of the two edges horizontally projected in *i* and *f*, and the other from those projected in *h* and *g*, and their diameters are parallel and equal to *if* and *hg*.

The groins spring from the upper ends of the four inner edges of the pillars, the one from those projected in *a* and *b*, and the other from those projected in *c* and *e*, and *ab* and *ce* are respectively equal and parallel to the transverse axes of the groins, the common conjugate axis being equal and parallel to the radius $k'k''$.

The planes of the outer faces of the pillars are produced upwards, inclosing the mass of masonry supported by the arch

To construct the perspective of the arch thus placed, let us take, Fig. 137, the plane of the picture coincident with the

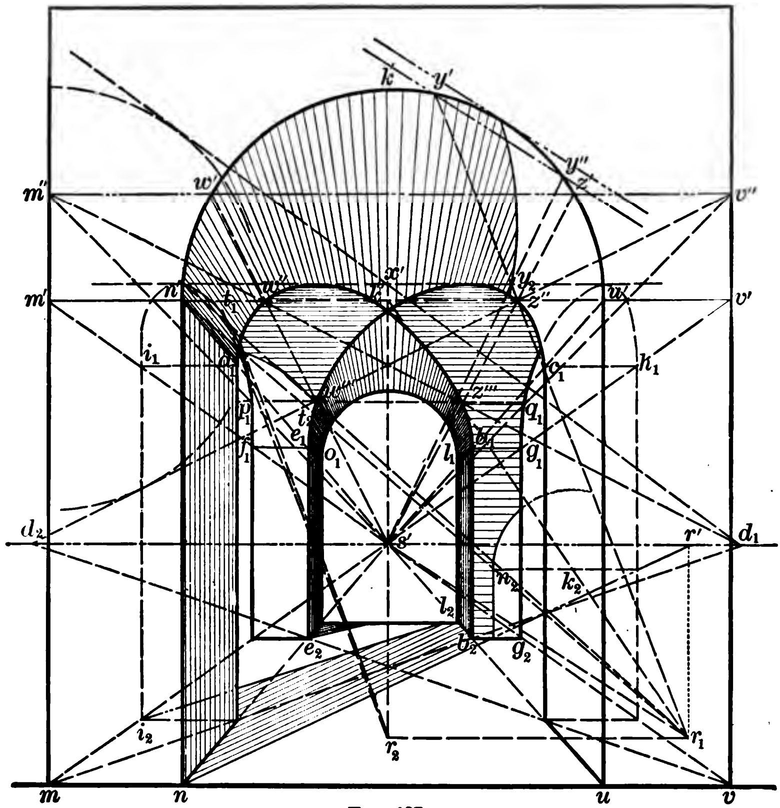

FIG. 137.

plane of the front faces of the front pillars, *mm'n'n* and *uu'v'v* being these front faces and their own perspectives, and let the principal point be at *s'* in a plane perpendicular to V and mid-way between the pillars.

The perspectives of the front pillars are constructed pre-

cisely as in Art. 295; a_1 and i_1 being the perspectives of the upper ends of the two back edges of the first, and c_1 and h_1 those of the second pillar; md_1, the perspective of the diagonal corresponding to mq in Fig. 136, intersects us', the perspective of the perpendicular corresponding to ul in b_2, which is the perspective of the lower end of the edge of the back pillar corresponding to b; b_2g_2, parallel to the ground line, is the perspective of the edge corresponding to bg, and b_2l_2 that corresponding to bl. Through b_2, g_2, and l_2 draw b_2b_1, g_2g_1, and l_2l_1 until they meet $u's'$ and $v's'$ in b_1, g_1, and l_1, and draw b_1g_1 and b_1l_1, thus completing the perspectives of the two faces of the back pillar which can be seen.

In the same way the perspective of the other back pillar may be constructed, e_2 being the perspective of the point corresponding to e.

On $n'u'$ as a diameter describe the semicircle $n'k'u'$; it is the front semicircle of the arch in the plane of the picture and its own perspective. The other base of the same cylinder being parallel to the plane of the picture will be in perspective a semicircle, and since it springs from the points of which o_1 and l_1 are the perspectives, o_1l_1 will be the diameter and the semicircle on this its perspective.

To obtain points of the perspectives of the groins and side circles, intersect the arch by horizontal planes. Each plane will cut from the cylinders rectilinear elements, the intersection of which will be points of the groins, and from the side faces of the arch, straight lines which will intersect the elements parallel to the plane of the picture in points of the side circles.

Let $m''v''$ be the trace of such a plane. It intersects the cylinder whose axis is perpendicular to V in two elements which pierce V at w' and z', vanish at s', and are represented by $w's'$ and $z's'$. The same plane intersects the side faces of the arch in lines represented by $m''s'$ and $v''s'$. The two diago-

nals through m'' and v'' intersect the elements in points of the groins (see Fig. 136). $m''d_1$ and $v''d_2$ are the perspectives of these diagonals intersecting $w's'$ in w'' and w''' and $z's'$ in z'' and z''', points of the perspectives of the groins. $w''z''$ and $w'''z'''$ are the perspectives of the elements cut from the cylinder whose axis is parallel to V. These are intersected by $m''s'$ and $v''s'$ in points p_1 and q_1 of the perspectives of the side circles.

The perspectives of the groins thus determined are drawn, one from a_1 to b_1, the other from c_1 to e_1, and the perspectives of the side circles, one from i_1 to f_1, and the other from h_1 to g_1.

Since all four of these curves cross the element of contour of the cylinder whose axis is parallel to the plane of the picture, their perspectives will be tangent to the perspective of this element. To determine it, through S pass a plane perpendicular to the axis of the cylinder; r_2s' is its trace. It cuts from the cylinder a circle and from the visual plane tangent to the cylinder a straight line tangent to the circle. Revolve this plane about r_2s' until it coincides with V. The center of the circle will be at m', and S at d_1. The semicircle described with m' as a center and $k''n'$ as a radius is the revolved position of the circle cut from the cylinder, and the tangent to it from d_1, the revolved position of the line cut from the tangent plane. This pierces V at x', a point of the perspective of the element of contour. The line through x' parallel to the ground line is the tangent to all the curves.

nr' and i_2r' are the perspectives of the indefinite shadows cast on H by the vertical edges of the left-hand pillar (Art. 295). The point in which nr' intersects b_2g_2 is the perspective of a point of the shadow of nn' on the front face of the back pillar. This line being parallel to this face, its shadow will be parallel to itself as also the perspective of the shadow, which is terminated at n_2, the perspective of the shadow cast by n' (Art. 293).

From this point the shadow is cast on this face by the front circle, and is the arc of a circle as is its perspective (Art. 285). The shadow of the radius $n'k''$ is parallel to itself, as also the perspective of this shadow passing through n_2 and terminated at k_2. The arc described with k_2 as a center and k_2n_2 as a radius is then the *perspective of the shadow of the front circle on the face.*

e_2r' is the perspective of the shadow cast on H by the edge represented by e_2e_1, only a small part of which lies in the limits of the drawing.

Points of the shadow cast by the front circle on the cylinder, of which it is the base, are determined by intersecting the cylinder by planes of rays perpendicular to the plane of the picture. The traces of these planes are parallel to $s'r_1$ (Art. 292), and each plane cuts the front circumference in a point casting the shadow and the cylinder in a rectilinear element, which is intersected by the ray through the point in its shadow.

Let $y'y''$ be the trace of such a plane. It intersects the cylinder in an element represented by $y''s'$, and the circumference in the point y', and y_2 is the perspective of the shadow. The perspective of the shadow begins at the point in which a trace parallel to $y''y'$ is tangent to the circle.

Points of the shadow cast by the side circle on the left, on the cylinder of which it is a base, may be found by intersecting the cylinder by planes of rays perpendicular to the side faces of the arch. Each plane intersects the circumference in a point casting the shadow, and the cylinder in a rectilinear element which is intersected by the ray through the point in a point of the shadow.

The intersections of these planes with the outer side face of the arch vanish at r_2 (Art. 291). Let r_2t_1 be the perspective of one of these intersections. The plane intersects the side circumference in a point of which t_1 is the perspective, and the

cylinder in an element (not shown). The intersection between this element and t_1r_1 is the perspective of the point of shadow. The perspective of the curve evidently begins at the point of tangency of a line through r_2 tangent to $i_1p_1f_1$.

307. Perspectives made on curved surfaces. Although in general the perspectives of objects are made as in the preceding articles on a plane, it is not unusual to make them upon curved surfaces.

Extended *panoramic views* are thus made upon a right cylinder with a circular base, the point of sight being in the axis, and the observer thus entirely surrounded by pictures of objects. Objects are also sometimes represented on the interior of a spherical dome.

In all cases the perspectives of points are constructed by drawing through the points visual rays, and finding the points in which these pierce the surface on which the representation is made. The constructions involve only the principles contained in Arts. 276, 277, and 281.

PART V

ISOMETRIC DRAWING

Preliminary Definitions and Principles

308. Definitions. Let three straight lines be drawn intersecting in a common point and perpendicular to each other. two of them being horizontal and the third vertical; as the three rectangular coördinate axes in space (*Analytical Geometry*), or the three adjacent edges of a cube; then let a fourth straight line be drawn through the same point, making equal angles with the first three, as the diagonal of a cube. If a plane be now passed perpendicular to this fourth line, and the straight lines and other objects be orthographically projected upon it, the projections are called *isometric*.

The three straight lines first drawn are the *coördinate axes;* and the planes of these axes, taken two and two, are the *coördinate planes*. The common point is the *origin*.

The fourth straight line is the *isometric axis*.

If A designate the origin, the coördinate axes are designated, as in Analytical Geometry, as the axes AX, AY, and AZ, the latter being vertical; and the coördinate planes, as the plane XY, XZ, and YZ, the first being horizontal and the other two vertical.

309. Principles. Since the coördinate axes make equal angles with one another and with the plane of projection, it is evident that their projections will make equal angles with one another, two and two; that is, angles of 120°. Hence, if any three straight lines, as Ax, Ay, and Az, be drawn through a point

as A, Fig. 138, making with one another angles of 120°, these may be taken as the *projections of the coördinate axes*, and are the *directrices* of the drawing.

It is further evident that if any equal distances be taken on the coördinate axes, or on lines parallel to any one of them, their projections will be equal to one another, since each projection will be equal to the distance itself into the cosine of the angle of inclination of the axes with the plane of projection (Art. 206).

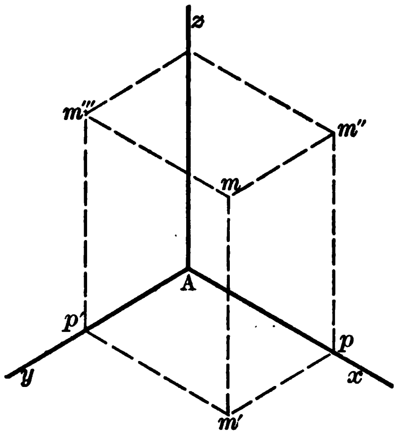

Fig. 138.

The angle which the diagonal of a cube makes with each adjacent edge is known to be 54° 44′; therefore the angle which any edge, or any of the coördinate axes, makes with the plane of projection will be the complement of this angle, viz. 35° 16′.

310. The isometric scale. If a scale of equal parts, Fig. 139, be constructed, the unit of the scale being the *projection* of any definite part of any coördinate axis, as *one inch*, *one foot*, etc., that is, *one inch multiplied by the natural cosine of* 35° 16′ = .81647 of an inch, we may, from this scale, determine the true length of the isometric projection of any given portion

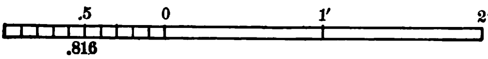

Fig. 139.

of any of the coördinate axes, or of lines parallel to them, by taking from the scale the same number of units as the number of inches, or feet, etc., in the given distance. *Conversely*, the true length of any corresponding line in space may be found by applying its projection to the isometric scale, and taking the

same number of inches, or feet, etc., as the number of parts covered on the scale.

311. Isometric drawing as distinguished from isometric projection. If, instead of laying off the foreshortened distances parallel to the isometric axes by means of the isometric scale, we should lay off the *true* distances by means of the common scale, the result would be an *isometric drawing*. Such drawings are much more convenient, both for the draftsman and the workman, and are the ones used in practice in preference to the isometric *projections*.

312. Advantages of isometric representation. Since in most of the framework connected with machinery and in the various kinds of buildings the principal lines to be represented occupy a position similar to that of the coördinate axes, viz. perpendicular to one another, one system being vertical, and two others horizontal, the isometric representation is used to great advantage.

A still greater advantage arises from the fact that in a drawing thus made, all lines parallel to the directrices are constructed on the same scale.

Isometric Representation of Points and Lines

313. Given the distance of a point from the three coördinate planes, to determine its position in an isometric drawing. — If a point be given, as in Analytical Geometry, by its coördinates, its position on the isometric drawing may be easily determined. Thus, Fig. 138, let Ax, Ay, and Az be the directrices, A being the projection of the origin. On Ax lay off Ap, equal to the distance of the point from the coördinate plane YZ. Through p draw pm' parallel to Ay, and make it equal to the distance of the point from the plane XZ. Through m' draw $m'm$ parallel to Az, and make it equal to the third given distance, and m will be the required projection.

314. To construct the isometric drawing of a straight line or curve. The isometric of a straight line parallel to any of the coördinate axes may be constructed by finding, as above, the isometric of one of its points, and drawing through this a line parallel to the proper directrix.

If the line is parallel to none of the axes, the isometric of two of its points may be found as above and joined by a straight line.

A curved line may be represented by determining the isometrics of a sufficient number of its points.

315. Isometric of a circle whose plane is parallel to one of the coördinate planes. If the circumference of a circle be in, or parallel to, one of the coördinate planes, its isometric may be constructed thus: At the extremities of the two diameters which are parallel to the coördinate axes, draw tangents, thus circumscribing the circle by a square. Each set of tangents will be parallel to the parallel diameter and tangent to the circle (Art. 61); hence these two equal diameters will be equal conjugate diameters of the ellipse which is the projection of the circle, and since these conjugate diameters are parallel to the directrices they will make with each other an angle of 120°. Upon these the ellipse may be described (Art. 257), taking care to make it tangent to the projections of the four tangents.

Practical Problems

316. Problem 96. **To construct the isometric drawing of a cube.**

Let the origin be taken at one of the upper corners of the cube, the base being horizontal, and let Ax, Ay, and Az, Fig. 140, be the directrices.

From A on the directrices lay off the distances Ax, Ay, and Az, each equal to the number of units of length in the edge of the cube. These will be the representations of the three edges

of the cube which intersect at A. Through x, y, and z, draw xe, xg, ye, yc, zc, and zg parallel to the directrices completing the three equal rhombuses Axey, etc. These will be the three faces of the cube which are seen, and the representation will be complete.

317. The ellipses constructed upon the equal lines kl, qs, st, uv, etc., Fig. 140, evidently represent the three circles inscribed in the squares which form the faces of the cube, and these lines are the equal conjugate diameters of the ellipses (Art. 315), lk being drawn through the middle point of Ax, and sq through the middle point of Az and parallel respectively to Az and Ax, m being the center of the circle.

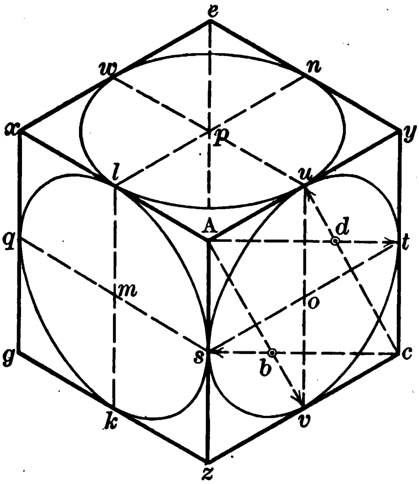

Fig. 140.

A very close approximation to the ellipses may be obtained with circular arcs, thus: With center c and radius cu, strike the arc us. With center A and radius At, strike the arc tv. Then with d and b as centers, strike the arcs ut and sv. Similarly for the other two ellipses.

318. Problem 97. **To construct the isometric drawing of an upright rectangular beam with its shade and shadow on the horizontal plane.**

Let the origin A and the directrices, Fig. 141, be taken as in the preceding problem.

On Ax lay off the distance Ax equal to the breadth of the beam; on Ay, its thickness, and on Az, its length, and complete

the three parallelograms $Axcz$, $Aybz$, and $Axey$. These will represent the three faces of the beam which are seen.

Let a' be assumed as the isometric of the point in which a ray of light through A pierces H; then Aa' will be the isometric of the ray, and za' the isometric of the shadow of the edge AZ (Art. 259). Through a' draw $a'y'$ parallel and equal to Ay. It is the shadow of the edge AY. $y'e'$, equal and parallel to ye, is the shadow cast by YE, and de' that of the edge DE.

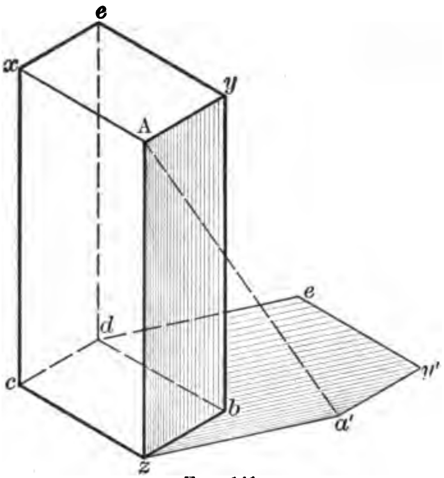

FIG. 141.

The face represented by $Azby$ is in the shade.

319. PROBLEM 98. **To construct the isometric drawing of the framework of a simple horizontal platform resting on four rectangular supports.**

Let A, Fig. 142, be the upper corner of one of the horizontal beams, the directrices being as in the preceding problems.

Lay off Ax, Ab, and Az equal respectively to the number of units in the length, breadth, and thickness of the horizontal beam, and complete the three parallelograms Aq, Ax', Ab', thus forming the representation of the beam. Lay off zc equal to the distance of the first support from the end of the beam, cd equal to its thickness, cc' parallel to Az equal to its height, and $c'e'$ parallel to Ay its breadth, and complete the parallelograms dc' and ce'. Lay off zf equal to the distance of the second support from the end of the beam, and complete it in the same way.

Lay off ba equal to the horizontal distance between the two

beams, and construct the drawing of the second beam and its supports precisely as the first was constructed.

From a to n lay off the distance from the end of the beam to the first cross-piece. Make nn' equal to its breadth, and np parallel to Az equal to its thickness, and draw lines through n, n',

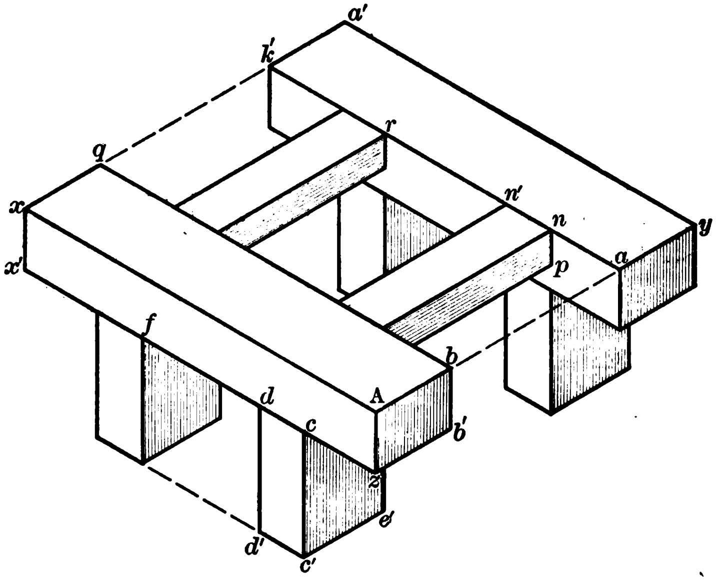

FIG. 142.

and p parallel to Ay. Lay off ar equal to the distance of the second cross-piece from the end of the beam, and construct it in the same way as the first.

The visible parts of the faces of the framework in the shade are darkened in the drawing.

320. PROBLEM 99. **To construct the isometric drawing of a house with a projecting roof.**

Let A, Fig. 143, be the upper end of the intersection of the front and side faces of the house, Ax, Ay, and Az the directrices.

On Ax lay off the length, on Ay the breadth of the house,

and on Az the height of the side walls, and complete the two parallelograms Ap and Ap'; they represent the side and front faces of the house.

At b, the middle point of Ay, draw br parallel to Az, and make it equal to the height of the ridge of the roof above AY,

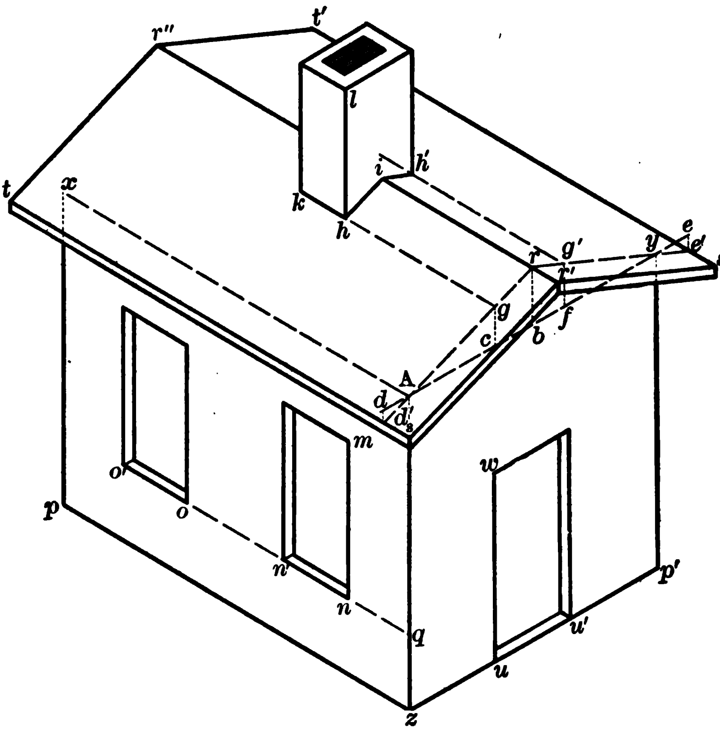

Fig. 143.

and join rA and ry; these lines will represent the intersections of the face ZY with the roof. From r lay off rr' equal to the distance that the roof projects beyond the walls of the house, and draw $r's$ and $r's'$ parallel respectively to rA and ry. On Ay lay off Ad and ye, each equal to rr', and draw dd' and ee' parallel to Az, intersecting Ar and ry produced in d' and e'. These

will be points of the eaves. Draw $r'r''$ parallel to Ax, and make it equal to the length of the ridge, and complete the parallelograms $r't$ and $r't'$. These will be the two inclined faces of the roof.

From b lay off bc and bf, each equal to the distance of the side faces of the chimney from the ridge, and draw cg and fg' parallel to Az, intersecting Ar and ry in g and g', and draw gh and $g'h'$ parallel to $r'r''$. They will be the intersections of the planes of the side faces of the chimney with the roof. Make gh equal to the distance of the front face of the chimney from the front face of the house, and draw hi and ih' parallel to Ar and ry; they will represent the intersection of the front face of the chimney with the roof. Make hk equal to the thickness of the chimney, and hl equal to its height, and complete its drawing as in the figure.

From z lay off zu equal to the distance of the door from the edge Az, uu' equal to its width, and uw equal to its height, and complete the parallelogram $u'w$. Make zq equal to the distance of the windows above the base, qn and qo their distances from the edge Az, nn' and oo' their width, and nm equal to their height, and complete the parallelograms.

321. A knowledge of the preceding simple principles and constructions will enable the draftsman to make isometric drawings of the most complicated pieces of machinery, and most extended collections of buildings, walls, etc.; drawings which not only present to the eye of the observer a very good representation of the objects projected, but are of great use to the machinist and builder.